INTRODUCTION TO FRACTIONAL FIELD THEORY

Ervin Goldfain

Advanced Technology and Sensor Group, Welch Allyn Inc., Skaneateles Falls, NY 13153

Abstract

As it is known, the Standard Model for particle physics (SM) has been successfully tested at all accelerator facilities and is currently the best tool available for understanding the phenomena on the subatomic scale. Conventional wisdom is that the SM represents only the low-energy limit of a more fundamental theory and that it can be consistently extrapolated to scales many orders of magnitude beyond the energy levels probed by the Large Hadron Collider (LHC).

Despite its impressive performance, the SM leaves out a fairly large number of unsolved puzzles. In contrast with the majority of mainstream proposals on how to address these challenges, the approach developed here exploits the idea that spacetime dimensionality becomes scale-dependent near or above the low TeV scale. This conjecture has recently received considerable attention in theoretical physics and goes under several designations, from "*fractional field theory*", "*continuous dimension*" to "*dimensional flow*" and *"dimensional reduction"*. Drawing from the principles of the Renormalization Group program, our key finding is that the SM represents a *self-contained multifractal set*. The set is defined on continuous spacetime having arbitrarily small deviations from four-dimensions ($\varepsilon = 4 - D << 1$), referred to as a "*minimal fractal manifold*" (MFM). The book explores the full dynamical implications of the MFM and, staying consistent with experimental data, it offers novel explanations on some of the unsolved puzzles raised by the SM.

"Rereading classic theoretical physics textbooks leaves a sense that there are holes large enough to steam a Eurostar train through them. Here we learn about harmonic oscillators and Keplerian ellipses - but where is the chapter on chaotic oscillators, the tumbling Hyperion? We have just quantized hydrogen, where is the chapter on the classical 3-body problem and its implications for quantization of helium? We have learned that an instanton is a solution of field-theoretic equations of motion, but shouldn't a strongly nonlinear field theory have turbulent solutions? How are we to think about systems where things fall apart; the center cannot hold; every trajectory is unstable?"

"Chaos: Classical and Quantum I: Deterministic Chaos "

- P. Cvitanovic *et al.*

(http://chaosbook.org/chapters/ChaosBook.pdf)

TABLE OF CONTENTS:

INTRODUCTION

This book develops a new perspective on the dynamical structure of the *Standard Model for particle physics* (SM), a framework that successfully explains the subatomic world and its fundamental interactions. The SM includes the $SU(3)\otimes SU(2)\otimes U(1)$ gauge model of strong and electroweak interactions along with the Higgs mechanism that spontaneously breaks the electroweak $SU(2)\otimes U(1)$ group down to the $U(1)$ group of electrodynamics. It has been confirmed countless times in all accelerator experiments, including the first round of runs at the LHC. The main motivation behind our work stems in the fact that, despite being overwhelmingly supported by experimental data, the SM has many puzzling aspects, such as the large number of physical parameters, a triplication of chiral families and the existence of three gauge interactions. Some of the unsettled issues revolve around the following questions:

- *Is the Higgs boson solely responsible for the electroweak symmetry breaking and the origin of mass?* The current view supports this assertion, although understanding of the Higgs sector remains widely open at this time [1, 2]. There are two primary mass-generation mechanisms in the SM: *the Higgs mechanism* of electroweak symmetry breaking, accounting for the spectrum of massive gauge bosons and fermions, and *dimensional transmutation*, partially responsible for the mass of baryonic matter. While technical aspects of both mechanisms are well under control, neither one is able to uncover the origin of the electroweak scale or of the Higgs boson mass.
- *Are fundamental parameters of the SM finely tuned?* The mass of the Higgs boson is sensitive to the physics at high energy scales. If there is no physics

beyond the SM, the elementary Higgs mass parameter must be adjusted to an accuracy order of 1 part in 10^{32} in order to explain the large gap between the TeV scale and the Planck scale [3].

- *What is the origin of quark, lepton and neutrino mass hierarchies and mixing angles?* These "flavor" parameters account for most of the basic parameters of the SM, and their pattern remains elusive. New particles at or above the TeV scale with flavor-dependent coupling charges are postulated in many scenarios, and observation of such particles would provide critical insights to these puzzles [4].
- *What is the physical nature and composition of Dark Matter and how is the SM related to the gravitational interaction?*
- *What is the underlying mechanism behind the matter-antimatter asymmetry in the Universe?*

It is generally believed that we are at a crossroads in the development of high-energy theory. Is there any compelling path to follow in our model-building efforts? We came a long way to recognize that, in general, Nature fails to fit the streamlined framework of conventional quantum field theories (QFT). Systems of quantum fields that are

- weakly interacting,
- nearly linear and stable against disturbances,
- perturbatively renormalizable,

form the backbone of "*effective*" QFT and are likely to represent exceptions rather than the rule. And yet we also know that both QFT and SM work exceptionally well up to the low TeV range probed by the LHC. A dilemma has undoubtedly surfaced on how to best proceed. For example, over the years, the many unsolved challenges of the SM led to an overflow of extensions targeting the physics beyond the SM scale. The majority of these proposals focus on solving some unsatisfactory aspects of the theory while introducing new unknowns. Experiments are expected to provide guidance in pointing to the correct theory yet, so far, LHC searches show no credible hint for physics beyond the SM up to a center-of-mass energy of $\sqrt{s}$ = 8 TeV [5]. These results, albeit entirely preliminary and subject to instant changes during the second LHC run, suggest two possible scenarios, namely:

- SM fields are either decoupled or ultra-weakly coupled to new dynamic structures emerging in the low or intermediate TeV scale,
- There is an undiscovered and possibly non-trivial connection between the SM and TeV phenomena.

It is often said that progress on the theoretical front requires understanding the first principles that drive Nature. The guiding principle we follow throughout this book is the *universal behavior of nonlinear dynamical systems*. We believe that there are reasons to conclude that this principle underlies a broad range of phenomena on the subatomic scale. In particular,

- The *universality principle* is a natural tool for decoding the dynamics of the SM, a manifestly nonlinear theory whose structure is based on self-interacting gauge and Higgs fields. As explained below, the principle also implies that spacetime dimensionality becomes scale-dependent near or above the low TeV scale. This conjecture has recently seen growing interest in theoretical physics and goes under several designations, from "*fractional field theory*" to "*continuous dimension*", *"non-integer metric"* and "*dimensional flow*". Drawing from the ideas of the Renormalization Group (RG) program, a key finding below is that the SM represents a *self-contained multifractal set*. The set is defined on continuous space-time having arbitrarily small deviations from four-dimensions, referred to as a "*minimal fractal manifold*" (MFM). Here we explore the dynamical implications of the MFM and, staying consistent with experimental data, we show that they offer novel insights on some of the unsolved puzzles raised by the SM.

- In contrast with many mainstream proposals, the universality principle hints that moving beyond the SM requires further advancing the RG program. In particular, understanding the nonlinear dynamics of RG flow equations and the transition from smooth to fractal dimensionality of spacetime are essential steps for the success of this endeavor. RG trajectories form a nonlinear and multidimensional system of coupled differential equations. The traditional assumption is that these equations describe parameter evolution towards *isolated* and *stable fixed points*. There is evidence today that this assumption is too restrictive, that it may ignore the rich dynamics of the flow in the presence of perturbations, in particular the emergence of *bifurcations, limit cycles* and *strange attractors* [6]. This may alter the conclusion (drawn from a linear stability analysis) that the flow is well-behaved and that non-renormalizable interactions become irrelevant at the electroweak (EW) scale.
- Our approach does not rely on additional hypotheses, symmetries or degrees of freedom beyond what the SM is based upon. It is also in line with the emerging science of complexity, in general, and to the well-developed fields of nonlinear

dynamics, fractal geometry and chaotic behavior, in particular. A key feature of the MFM is that the assumption $\varepsilon = 4 - D$ << 1, postulated near the EW scale, is the *only sensible way* of asymptotically matching all consistency requirements mandated by relativistic QFT and the SM [7-9, 15]. In particular, large departures from four-dimensionality imply non-differentiability of space-time trajectories in the conventional sense. This in turn, spoils the very concept of "speed of light" and it becomes manifestly incompatible with the Poincaré symmetry.

Few words of caution are now in order, namely,

- It must be emphasized at the outset that ideas discussed here stand in sharp contrast with the multitude of avenues followed by Quantum Gravity theories such as, but not limited to, String/M theories, Supergravity, Loop Quantum Gravity, Deformed Special Relativity, Spin Foam models of quantum spacetime, Black Hole phenomenology, Deformed Special Relativity, Causal Dynamical Triangulation, Poincaré Invariant Networks, Tensor Networks, Causal Sets, Lorentz Invariance Violation, Horava-Lifschitz gravity, Asymptotic Safety, Planck scale phenomenology and so on [14]. The path taken here *does not advocate any changes* to either General or Special Relativity or the current framework of the SM. It does not substitute the spacetime fabric with discrete networks of interconnected entities. Rather, our work may be remotely tied to the study of wavefunction multifractality and multifractal behavior of disordered quantum systems in condensed matter applications [10, 11]. Moreover, the low-level fractal topology described by the MFM may be associated with the upper boundary of *q*-deformed Quantum Field Theory, that is, $q \to 1 \Rightarrow \varepsilon = 0$ [12, 13].

- By default, given the breadth and complexity of topics linked to the development of QFT and SM, our book cannot claim to be either fully rigorous or formally complete. The sole intent here is to proceed from a less conventional standpoint and outline a new research strategy. Many premises and consequences of our approach are left out to avoid excessive information. Ideas are introduced in the simplest possible context with the caveat that they can be further extended to more realistic scenarios. For concision and simplicity, the mathematical presentation is kept at an elementary level.

The outline of the book is as follows: the basics of regularization theory as key tool of the RG program are discussed in the first section. This sets the stage for section 2, where we argue that the continuum limit of QFT is a weak manifestation of fractal geometry. Nonlinear dynamics of RG flow equations and their ability to account for the self-similar structure of SM parameters form the object of section 3. Drawing on these premises, section 4 argues that, near the electroweak scale, the ordinary four-dimensional spacetime turns into a MFM and that the SM can be understood as a self-contained multi-fractal set. Along the same line of inquiry, section 5 shows that the MFM can account for the dynamic generation of mass scales in QFT. Next couple of sections cover several features of the MFM that are also relevant to QFT and the physics of the SM, namely, charge quantization and the topological underpinning of quantum spin. Casting the MFM as asymptotic embodiment of non-commutative field theory forms the topic of section 8.The subtle duality between the MFM and classical gravity is touched upon in section 9. To provide proper guidance to the main text, several Appendix sections are introduced at the end of the book.

The reader is urged to keep in mind the introductory nature of this work. Further research and independent experimental validation are needed to substantiate, refute or develop the body of ideas outlined here.

1. BASICS OF REGULARIZATION THEORY

As it is known, the technique of regularization assumes that divergent quantities of perturbative QFT depend on a *continuous regulator* η [1]. The regulator can be either a large cutoff $\eta = \Lambda_{UV}$ or an infinitesimal deviation of the underlying space-time dimension, viz. $\eta = \varepsilon << 1$, $D \to D - \varepsilon$. A divergent quantity O becomes a function of the regulator, $O = O(\eta)$, asymptotically approaching the original quantity in the limit $\eta^{-1} = \Lambda_{UV}^{-1} \to 0$ or $\eta = \varepsilon \to 0$. As a result, in close proximity to this limit, the quantity of interest is no longer singular ($|O(\eta)| < \infty$). To fix ideas, consider the one-loop momentum integral of the massive φ^4 theory defined on a two-dimensional Euclidean space-time ($D = 2$)

$$\Sigma = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + m^2} \tag{1.1}$$

The integral is logarithmically divergent at large momenta $\Sigma(p^2) \to \infty$ for $p \to \infty$. One way to regularize (1.1) is to upper-bound it with a sharp *momentum cutoff* $\Lambda_{UV} >> m$ as in

$$\Sigma_c = \int_0^{\Lambda_{UV}^2} \frac{dp^2}{4\pi} \frac{1}{p^2 + m^2} = \frac{1}{4\pi} \ln(\frac{\Lambda_{UV}^2 + m^2}{m^2}) \tag{1.2}$$

The *Pauli-Villars regularization* method is based on subtracting from (1.1) the same integral having a larger momentum scale $\Lambda >> m$, that is,

$$\Sigma_{PV} = \int \frac{d^2 p}{(2\pi)^2} (\frac{1}{p^2 + m^2} - \frac{1}{p^2 + \Lambda^2}) = \frac{1}{4\pi} \ln(\frac{\Lambda^2}{m^2}) \tag{1.3}$$

By contrast, *dimensional regularization* posits that the spacetime dimension can be analytically continued to $D - \varepsilon$, which turns (1.1) into

$$\Sigma_{DR} = \mu^{\varepsilon} \int \frac{d^{2-\varepsilon} p}{(2\pi)^{2-\varepsilon}} \frac{1}{p^2 + m^2} \tag{1.4}$$

where μ is an arbitrary mass scale that preserves the dimensionless nature of Σ_{DR} (1.4) can be formulated as [1- 4]

$$\Sigma_{DR} = \frac{1}{4\pi} [\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - \ln(\frac{m^2}{\mu^2}) + O(\varepsilon)] \tag{1.5}$$

in which γ stands for the Euler constant. Comparing (1.3) with (1.5) and further taking μ to be on the same order of magnitude with m ($\mu = O(m)$) leads to the identification

$$\frac{1}{\varepsilon} \sim \ln(\frac{\Lambda^2}{m^2}) \tag{1.6}$$

Side by side evaluation of (1.2) and (1.5) gives instead [1]

$$\frac{\mu^2}{\Lambda_{UV}^2} \approx \frac{m^2}{\Lambda_{UV}^2} \sim \frac{1}{e^{\frac{2}{\varepsilon} - \gamma}} = O(\varepsilon) \tag{1.7}$$

Relations (1.6) and (1.7) describe the same scaling behavior if the dimensional parameter is assumed to be vanishingly small (ε << 1) and $m = O(\mu)$ << $\Lambda = O(\Lambda_{UV})$. From these considerations we develop the reasonable numerical approximation

$$\boxed{\varepsilon \sim \frac{m^2}{\Lambda_{UV}^2}} \tag{1.8}$$

We'll make use of (1.8) in the sections 3 and 4.

An important observation is now in order. The generating functional describing the physics at an arbitrary observation scale μ in exactly four dimensions is given by the path integral [3]

$$Z_\mu[j_\nu] = \int D\varphi \exp\{-d^4x[L_0(\varphi,\mu) + L_{\text{int}}(\varphi,\mu) + \int d^4(x)\, j_\nu(x)\varphi(x)]\} \tag{1.9}$$

A drawback of dimensional regularization is that, unlike the RG prescription used in the momentum cutoff scheme, it cannot be extrapolated beyond perturbation theory. There is no realistic way of replicating the path integral (1.9 in non-integer dimensions (that is, $\varepsilon \neq 0$)) whereby the dynamics can be specified by an effective Lagrangian expanded in local operators [3]. Therefore, a non-perturbatively valid construction of a local QFT rooted in dimensional regularization appears to be impossible. Fortunately, as we argue throughout the book, introducing the MFM as space-time endowed with *arbitrarily small* deviations from four dimensions (ε << 1), provides *the only sensible* solution of working in a region that asymptotically matches the conditions mandated by local QFT and the SM.

2. QUANTUM FIELD THEORY AS WEAK MANIFESTATION OF FRACTAL GEOMETRY

We discuss in this section two theoretical arguments suggesting that the continuum limit of QFT leads to fractal geometry. The first argument stems from the Path Integral formulation of QFT, whereas the second one is an inevitable consequence of the Renormalization Group (RG).

2.1 QFT AS CRITICAL BEHAVIOR IN STATISTICAL PHYSICS

A basic task in perturbative QFT is to compute the time-ordered n-point Green function, i.e. [1]

$$\langle 0|T\{\varphi(x_1)\varphi(x_2)...\varphi(x_n)\}|0\rangle = \frac{\int D\varphi\,\varphi(x_1)\varphi(x_2)...\varphi(x_n)e^{iS}}{\int D\varphi\, e^{iS}} \tag{2.1}$$

Performing the rotation to Euclidean space $e^{iS} = e^{-S_E}$ and taking the above integral to run over all configurations that vanish as the Euclidean time goes to infinity ($t_E = \pm\infty$), leads to the conclusion that (2.1) is formally identical to the correlation function of classical statistical systems. A natural question is then: *What kind of statistical system is able to duplicate the properties of a QFT described by (2.1)?*

In order to compute (2.1), it is convenient to discretize the Euclidean space using, for example, a four-dimensional lattice with constant spacing δ. Under the assumption that the number of lattice sites is finite, the path integral of (2.1) becomes well defined and the question posed above amounts to taking the *continuum limit* $\delta \to 0$ at the end of calculations.

To fix ideas, consider the two-point Green function for a massive field theory defined on four-dimensional spacetime with Euclidean metric δ_{μ}^{ν}

$$\langle \varphi(x)\varphi(0)\rangle = \int \frac{d^4 p}{(2\pi)^4}\frac{\exp(ipx)}{p^2+m^2} \tag{2.2}$$

with $|p|^2 = p_\mu p^\mu$ and $px = p_\mu x^\mu$. Calculations are considerably simplified if $m|x| >> 1$, in which case (2.2) becomes

$$\langle \varphi(x)\varphi(0)\rangle \sim \frac{1}{|x|^2}\exp(-m|x|) \tag{2.3}$$

Expressing the space-time separation as $|x| = n\delta$ and assuming $n >> 1$ leads to

$$\langle \varphi(x)\varphi(0)\rangle \sim \exp(-n\delta m) \tag{2.4}$$

By analogy with statistical physics, the behavior of

$$\langle \varphi(x)\varphi(0)\rangle \sim \exp(-\frac{n}{\xi}) \tag{2.5}$$

determines the dimensionless correlation length ξ. Comparing (2.4) and (2.5) yields

$$\xi = \frac{1}{\delta m} \tag{2.6}$$

It is immediately apparent that the continuum limit $\delta \to 0$ of the massive theory (that is, for $m \neq 0$) implies singular correlation length, that is, $\xi \to \infty$. This conclusion shows that QFT models phenomena that are strikingly similar with the ones describing critical behavior in statistical physics. Since all phenomena near criticality are scale-free and lay on a fractal foundation [2], it is clear that *the continuum limit of QFT necessarily leads to fractal geometry*.

2.2 RG AND THE ONSET OF SELF-SIMILARITY IN QFT

As it is known, the RG studies the evolution of dynamical systems scale-by-scale as they approach criticality [2, 3]. It does so by defining a mapping between the observation scale (denoted by μ) and the distance ($x = |\mu - \mu_c|$) from the critical point, where the passage $x \to 0$ defines the *continuum limit in energy space*. The universal utility of the RG is based on the existence of self-similarity of all observables as $x \to 0$.

To illustrate this point, consider a generic model whose fields are evenly distributed on the discrete lattice of points. The behavior of the Lagrangian $L(x)$ in the RG formalism is given by the following set of transformations [2]

$$x' = \sigma(x) \tag{2.7}$$

$$L(x) = h(x) + \frac{1}{\Delta} L[\sigma(x)] \tag{2.8}$$

Here, Δ is a constant describing the rescaling of the Lagrangian upon shifting the scale to the critical value ($\mu \to \mu_c$), the function $\sigma(x)$ is called the *flow map* and

$$L(x) = L(\mu) - L(\mu_c) \tag{2.9}$$

such that $L(x)=0$ at the critical point. The function $h(x)$ represents the non-singular part of $L(x)$. Assuming that both $L(x)$ and $\sigma(x)$ are differentiable, the critical points are defined as the set of values at which $L(x)$ becomes singular, that is, when $\frac{dL}{dx}\to\infty$.

Then, the formal solution of (2.8) can be presented as the recursive sequence

$$f_0(x)=h(x) \tag{2.10}$$

$$f_{n+1}(x)=f_0(x)+\frac{1}{\Delta}f_n\left|\sigma(x)\right|,\quad n=0,1,2.... \tag{2.11}$$

where

$$f_n(x)=\sum_{i=0}^{n}\frac{1}{\Delta^i}h\,[\sigma^{(i)}(x)] \tag{2.12}$$

Here, the superscripts (i) denote composition, that is,

$$\sigma^{(2)}=\sigma\left[\sigma(x)\right],\,\sigma^{(3)}=\sigma\left[\sigma^{(2)}(x)\right]... \tag{2.13}$$

The renormalized Lagrangian assumes the form

$$L(x)=\lim_{n\to\infty}f_n(x) \tag{2.14}$$

The above relation indicates that *all copies of the Lagrangian specified by the iteration index* n *become self-similar in the limit* $n\to\infty$. Furthermore, if x designates a generic coupling constant ($x=g(\mu)$) whose critical value occurs at $g_c=g(\mu_c)$, the Lagrangian

$$L(g)=\sum_{n=0}^{\infty}\frac{1}{\Delta^n}h\left[\sigma^{(n)}(g)\right] \quad (2.15)$$

may be shown to become singular at $g=g_c$. In the neighborhood of $g=g_c$ (2.15) follows a power law that is typical for the onset of fractal behavior, namely:

$$L(g)=(const).(g-g_c)^{\rho} \quad (2.16)$$

where ρ stands for the critical exponent.

This brief analysis clearly points out that QFT is a *hidden manifestation of fractal geometry*. As we have repeatedly shown over the years, exploiting the fractal underpinnings of QFT and RG may provide viable solutions for the many puzzles associated with the SM [4 - 9].

3. NONLINEAR DYNAMICS OF THE RG FLOW AND SM PARAMETERS

Previous section has surveyed the close connection between fractal geometry, critical phenomena and the RG treatment of QFT. In statistical physics, the divergence of the correlation length near a second-order phase transition signals that the properties of the critical point are insensitive to the microscopic details of the system. Likewise, the approach to conformal point in effective QFT is considered to be insensitive to the physics of the ultraviolet (UV) sector, according to the so-called *cluster decomposition principle* [1, 2]. One is therefore motivated to search for a description of critical behavior applicable to a wide range of phenomena, from many-body statistical systems to interacting quantum fields. As we argue below, the *Landau-Ginzburg-Wilson* (LGW) model offers a sound baseline for such an enterprise.

To drive home the main point, in this section we restrict our analysis to the infrared (IR) sector of the self-interacting scalar field theory. It is in this limit where the LGW model provides a unified description of the long-wavelength behavior associated with many dynamical systems [3]. Despite the fact the LGW model is not a realistic substitute for relativistic QFT and the SM, it gives valuable insight into how dynamics evolves near criticality. With these cautionary remarks in mind, the LGW model provides an effective benchmark for understanding the primary attributes of IR quantum electrodynamics (QED) or UV quantum chromodynamics (QCD) and asymptotically free theories.

This section is divided into two parts. In paragraph 3.1 we introduce the mapping theorem which establishes a useful analogy between scalar field theory and the IR sector of the Yang-Mills theory. Next paragraph develops the nonlinear dynamics of RG flow equations which are found to provide a straightforward explanation on the hierarchical pattern of SM parameters.

3.1 THE MAPPING THEOREM

The electroweak group of the SM is represented by $SU(2)\otimes U(1)$ and is broken at a scale approximately given by $M_{EW}=O(G_F^{-1/2})$, in which G_F is the Fermi constant [1, 2]. Yang-Mills fields associated with $SU(2)$ are vectors denoted as $A_\mu^a(x)$, in which $\mu=0,1,2,3$ is the Lorentz index and $a=1,2,3$ is the group index. To manage the large number of equations derived from the Yang-Mills theory, it is desirable to devise a method whereby $A_\mu^a(x)$ are reduced to analog fields having less complex structure. The *mapping theorem* allows for such a convenient reduction [4]. The action functional of classical scalar field theory in four-dimensional spacetime is defined as

$$S[\Phi]=\int d^4x[\frac{1}{2}(\partial\Phi)^2-\frac{1}{4!}g^2\Phi^4] \quad (3.1)$$

An extremum of (3.1) is also an extremum of the $SU(2)$ Yang-Mills action provided that:

a) g represents the coupling constant of the Yang-Mills field,

b) some components of $A^a_\mu(x)$ are chosen to vanish and others to equal each other.

In the most general case, the following approximate mapping between Yang-Mills fields and scalar $\Phi(x)$ holds [4]:

$$A^a_\mu(x)=\eta^a_\mu\Phi(x)+O(\frac{1}{\sqrt{2}g}) \quad (3.2a)$$

where η^a_μ are properly chosen constants. Mapping becomes exact in the Lorenz gauge $\partial^\mu A^a_\mu(x)=0$ and in the IR regime of strong coupling ($g\to\infty$).

$$A^a_\mu(x)\to\Phi(x)\Leftrightarrow\partial^\mu A^a_\mu(x)=0\,,\ \ g\to\infty \quad (3.2b)$$

3.2 DYNAMICS OF RG FLOW EQUATIONS

We start from the standard LGW action for the massive *O(N)* field theory in 3+1 dimensions in the presence of external sources [3]. It has a similar structure as (3.1) and is given by

$$S[A]=\int d^4x\{\frac{1}{2}A^a(x)[r-\Delta]A^a(x)+\frac{u}{4}[A^a(x)A^a(x)]^2-j^a(x)A^a(x)\}+S_{J_0} \quad (3.3)$$

Here, $A(x) = (A^a(x))$ represents the Yang-Mills field, $j = (j^a(x))$ is the external fermion current (whose contribution to the action in the absence of interactions is denoted by S_{J_0}). The summation convention is implied and the Lorentz index is omitted for simplicity. To make the derivation more transparent and without a significant loss of generality, we proceed with the following set of simplifying assumptions:

A3.1) the LGW model is placed on a MFM characterized by a spacetime dimension arbitrarily close to four, that is, $D = 4 - \varepsilon$, where $\varepsilon << 1$. According to the philosophy of critical phenomena in continuous dimension, ε is regarded as the sole control parameter driving the dynamics of the model [5]. With reference to (1.8), fine-tuning the dimensional parameter ε is formally equivalent to applying continuous changes to the momentum cutoff Λ_{UV}. The passage to the classical limit $\varepsilon \to 0$ can be approached in two separate ways:

1) $\Lambda_{UV} \to \infty$ and $0 < m << \infty$;

2) $\Lambda_{UV} < \infty$ and $m \to 0$.

The latter condition matches the infrared behavior of the LGW model, i.e. its long-wavelength properties $|Q| = O(m) = O(\varepsilon)$, in which $|Q|$ stands for the magnitude of momentum transfer. We exclusively focus below on this asymptotic regime, whereby $m \sim \varepsilon > 0$.

Both limits 1) and 2) are disfavored by our current understanding of the far UV and the far IR boundaries of field theory (see e.g. [2]). Theory and experimental data alike tell us that the notions of infinite *or* zero energy are, strictly speaking, meaningless. This is to say that either infinite energies (point-like objects) or zero energy (infinite distance scales) are *unphysical idealizations*. Indeed, there is always a finite cutoff at both ends of either energy or energy density scale (far UV = Planck scale, far IR = finite radius of the observable Universe or the non-vanishing energy density of the vacuum set by cosmological constant). These observations are also consistent with the estimated infinitesimal (yet non-vanishing) photon mass, as highlighted in [6, 7].

A3.2) In light of the mapping theorem introduced in section 4.1, the discussion is limited to the *O (1)* model, i.e. the gauge field is treated as a scalar.

A3.3) the overall fermion current contains two terms,

$$J(x) = j(x) + J_0(x) \tag{3.4}$$

where $j(x)$ represents he component that couples to $A(x)$ and $J_0(x)$ the free (non-interacting) component. If $j(x)$ is uniform, its contribution to the action may be presented as

$$S_j = -j\int A(x)d^dx = -jA_0 \tag{3.5}$$

Likewise, if we further assume that $J_0(x)$ is uniform as well, its contribution to the action is well approximated by an additive constant, that is [3],

$$S_{J_0} \sim J_0\int d^3x \sim J_0\mu^3 = J_0 O(m^3) \tag{3.6}$$

The action functional assumed the familiar form

$$S[A] = \int d^4x \{ \frac{1}{2} A(x)[r - \Delta] A(x) + \frac{u}{4}[A(x)]^4 - j(x)A(x)\} + S_{J_0} \quad (3.7)$$

A3.4) Section 3.1 has pointed out the close analogy between quantum field theory (QFT) and statistical systems near criticality. On this basis, we assume that the Yang-Mills model is reasonably well approximated by the LGW theory of critical behavior.

A3.5) It follows from A3.4) that the dimensional parameter of LGW theory and dimensional regulator of Yang-Mills theory $\varepsilon = 4 - D$ are identical entities. This identity is made explicit in the first row of Tab. 1 below.

A3.6) As stated above, we focus on the IR regime of Yang-Mills theory in which $M_{EW} = G_F^{-1/2}$ stands for the EW scale, G_F for the Fermi constant $\mu = O(m)$ for the running scale and the ultraviolet (UV) scale $\Lambda = \Lambda_{UV} > \mu > M_{EW}$ for the cutoff.

A3.7) The UV cutoff is not uniquely determined but smeared out by the contribution of high-energy noise [8]. The UV cutoff spans a range of values

$$\Lambda_{UV} \in \delta\Lambda_{UV} \quad (3.8)$$

(3.7) implies that, at any given μ and Λ_{UV}, dimensional parameter ε falls in the range

$$|\delta\varepsilon| = 2\mu \frac{\delta\Lambda_{UV}}{\Lambda_{UV}} \quad (3.9)$$

Elaborating from these premises leads to the following side-by-side comparison between the parameters of LGW of statistical physics and Yang-Mills theory:

Landau –Ginzburg -Wilson theory	**Yang-Mills theory**
Dimensional parameter ($\varepsilon = 4 - D$)	Dimensional regulator ($\varepsilon = 4 - D$)
Momentum cutoff (Λ)	Ultraviolet cutoff (Λ_{UV})
Temperature (T)	Energy scale ($M_{EW} < \mu < \Lambda_{UV}$)
Critical temperature (T_c)	EW scale (M_{EW})
Temperature parameter (r)	Deviation from the EW scale ($\delta\mu = \mu - M_{EW}$)
Coupling parameter (u)	Coupling constant (g^2)
External field (h)	Fermion current (j)

Tab. 1: Comparison between LGW of statistical physics and Yang-Mills theory

Under these circumstances, RG flow equations for $r = \delta\mu$, $u = g^2$ and fermion current $j = j_f$ read, respectively [3]

$$\frac{\partial(\delta\mu)}{\partial t} = (\delta\mu)(2 + bg^2) + ag^2$$

$$\frac{\partial g^2}{\partial t} = \varepsilon g^2 - 3b(g^2)^2 \tag{3.10}$$

$$\frac{\partial j_f}{\partial t} = (3 - \frac{\varepsilon}{2}) j_f$$

Here,

$$a = 3K_4\Lambda_{UV}^2, \qquad b = 3K_4, \qquad K_4 = (8\pi^2)^{-1} \tag{3.11}$$

On account of (3.7), the Wilson-Fisher (WF) fixed point of (3.10) is defined by the pair

$$(\delta\mu)^* = -\frac{a}{6b}\varepsilon \tag{3.12a}$$

$$(g^2)^* = \frac{\varepsilon}{3b} \tag{3.12b}$$

(3.12) acts as a non-trivial attractor of the RG flow. Because it resides on the critical line $\mu = M_{EW}$, it describes by definition a *massless* field theory ($r = \delta\mu = 0$) [3]. The non-vanishing field vacuum at the WF point results from minimization of (3.7), that is,

$$\mathrm{v}^* = \pm\sqrt{\frac{6(-\delta\mu)^*}{(g^2)^*}} = \pm 3(K_4)^{1/2}\Lambda_{UV} \tag{3.13}$$

(3.12) and (3.13) reveal how massive gauge bosons emerge at the WF point from critical behavior near $D = 4$. Let $\mathrm{v}^* = M$ denote the mass acquired by the gauge boson. Combining (3.11), (3.12) and (3.13) yields

$$\boxed{(g^2)^* M^2 = M_{EW}^2 = const.}$$

(3.14)

$$\boxed{(g^*)^2 \sim m_f^* \sim \varepsilon}$$

in which $m_f^* = O(j_f)$ stands for the normalized fermion mass [3]. On account of assumption (3.7), the WF attractor (3.12) changes from a *single isolated* point to a *distribution* of points.

It is important to note that (3.14) refers to quantities that are either nearly diverging or nearly vanishing. They may be understood as "*primitive*" observables of the MFM, replicating to some extent the concept of "*bare*" parameters of the RG theory. Echoing the case of RG, the passage from "*primitive*" to "*measured*" observables requires multiplication with a suitable normalization scale.

Our next step is to explore the link between the structure of the WF attractor and the parameters of SM.

3.3 WILSON-FISHER ATTRACTOR AS SOURCE OF PARTICLE MASSES AND GAUGE CHARGES

We are now ready to analyze the dynamics of (3.10) using the standard methods employed in the study of nonlinear systems [9]. To this end, we first note that the last equation in (3.10) is uncoupled to the first two. This enables us to reduce (3.10) to a planar system of differential equations. We next cast (3.10) in the form of a two-dimensional map, namely

$$(g^2)_{n+1} = (1+\varepsilon\Delta t)(g^2)_n - 3b\Delta t(g^2)_n \qquad (3.15)$$

$$(\delta\mu)_{n+1} = (\delta\mu)_n[1+2\Delta t + b\Delta t(g^2)_n] + a\Delta t(g^2)_n \qquad (3.16)$$

where Δt represents the increment of the sliding scale. Linearizing (3.15, 3.16) and computing its Jacobian J gives

$$J = 1 + (2 + \varepsilon)\Delta t > 1 \tag{3.17}$$

It follows that the map (3.15, 3.16) is dissipative for $\varepsilon \neq 0$ and asymptotically conservative in the limit $\varepsilon = \Delta t = 0$. Invoking universality arguments [10] we conclude that, near criticality, (3.15, 3.16) shares the same universality class with the quadratic map. Furthermore, in the neighborhood of the Feigenbaum attractor, ε approaches $\varepsilon_\infty = 0$ according to:

$$\varepsilon_n - \varepsilon_\infty \approx a_n \cdot \overline{\delta}^{-n} \tag{3.18}$$

Here, $n >> 1$ is the index counting the number of cycles generated through the period doubling cascade, $\overline{\delta}$ is the rate of convergence (in general, different from Feigenbaum's constant for the quadratic map) and a_n is a coefficient which becomes asymptotically independent of n, that is, $a_\infty = a$ [10, 11]. Substituting (3.18) in (3.14) yields

$$P_j(n) = \left[M_n^{-2} \quad (g^*)_n^2 \quad (m_f^*)_n \right] \propto \overline{\delta}^{-n} \quad \text{if} \quad n >> 1 \tag{3.19}$$

in which $j = 1, 2, 3$ indexes the three entries of (3.19). Period-doubling cycles are characterized by $n = 2^p$, with $p >> 1$. The ratio of two consecutive terms in (3.19) is then given by

$$\boxed{\frac{P_j(p+1)}{P_j(p)} = O[\overline{\delta}^{-(2^p)}]} \tag{3.20}$$

Numerical results derived from (3.20) are displayed in Tab. 2. This table contains a side-by-side comparison of estimated versus actual mass ratios for charged leptons and quarks and a similar comparison of coupling strength ratios. The best-fit rate of convergence is $\bar{\delta}=3.9$ which falls close to the numerical value of the Feigenbaum constant corresponding to hydrodynamic flows [11]. (3.19) and (3.20) imply that there is a series of terms containing massive electroweak bosons, namely

$$(M_n g_n^*)^2 = (M_{n+1} g_{n+1}^*)^2 = ... = (M_{n+q} g_{n+q}^*)^2 = ... = const. \tag{3.21}$$

For the first two terms of this series we obtain

$$\frac{M_Z^2}{M_W^2} = \frac{g_2^2 + e^2}{g_2^2} = 1 + \frac{\alpha_{EM}}{\alpha_2} \tag{3.22}$$

in which $\alpha_{EM} = e^2/4\pi$ is the electromagnetic coupling strength and $\alpha_2 = g_2^2/4\pi$ the strength of the weak interaction. The rationale for (3.22) lies in the fact that the charged gauge boson $W^{\pm}$ carries a superposition of weak and electromagnetic charges, whereas the neutral gauge boson Z^0 carries only the weak isospin charge. Inverting (3.22) and taking into account the last rows of Table 2, leads to

$$\frac{M_W^2}{M_Z^2} = \frac{1}{1+\frac{\alpha_{EM}}{\alpha_2}} = \frac{1}{1+\frac{1}{\bar{\delta}}} \approx 1 - \frac{1}{\bar{\delta}} = \cos^2\theta_W \tag{3.23}$$

(3.23) suggests a natural explanation for the Weinberg angle θ_W. Likewise, we may write (3.22) as

$$\frac{g_2^2}{M_W^2} = \frac{g_2^2 + e^2}{M_Z^2} = const \qquad (3.24)$$

This relation offers a fresh perspective on both Fermi constant and the vacuum expectation value of the Higgs boson. Indeed, in SM we have [12]

$$\frac{g_2^2}{M_W^2} = 4\sqrt{2} G_F \qquad (3.25)$$

and

$$V(\varphi^0) \propto \sqrt{\frac{1}{G_F \sqrt{2}}} \approx 246.22\ GeV \qquad (3.26)$$

where $V(\varphi^0)$ denotes the vacuum expectation value for the neutral component of the Higgs doublet.

A similar analysis may be carried out for *neutrinos*. Since neutrino oscillation experiments are only sensitive to neutrino mass squared differences and not to the absolute neutrino mass scale denoted by (m_ν^0), they can only supply lower limits for two of the neutrino masses, that is, $(m_{ATM}^2)^{1/2} \approx 5\times10^{-2}$ eV and $(m_{SOL}^2)^{1/2} \approx 1\times10^{-2}$ eV [13]). As a result, it is more relevant to consider experimentally constrained bounds on m_ν^0 reported from beta decay, neutrinoless double beta decay as well as from cosmological observations.

Based on these inputs, it makes sense to set the upper (U) and lower (L) limit values for the absolute neutrino mass scale as $(m_\nu^0)_U = 2$ eV and $(m_\nu^0)_L = 0.1$ eV. According to Tab. 2, ratios of charged lepton masses scale as $\bar{\delta}^{-2}$ and $\bar{\delta}^{-4}$, which suggests that m_ν^0 should naturally follow a $\bar{\delta}^{-8}$ or $\bar{\delta}^{-16}$ pattern . Table 3 displays a side-by-side comparison on the neutrino to electron mass ratio for $(m_\nu^0)_U$ and $(m_\nu^0)_L$, respectively, and shows that numerical predictions line up fairly well with current observations.

Parameter ratio	**Behavior**	**Actual**	**Predicted**
m_u / m_c	$\bar{\delta}^{-4}$	3.365×10^{-3}	4.323×10^{-3}
m_c / m_t	$\bar{\delta}^{-4}$	3.689×10^{-3}	4.323×10^{-3}
m_d / m_s	$\bar{\delta}^{-2}$	0.052	0.066
m_s / m_b	$\bar{\delta}^{-2}$	0.028	0.066
m_e / m_μ	$\bar{\delta}^{-4}$	4.745×10^{-3}	4.323×10^{-3}
m_μ / m_τ	$\bar{\delta}^{-2}$	0.061	0.066
M_W / M_Z	$(1-\frac{1}{\bar{\delta}})^{1/2}$	0.8823	0.8921
$(\alpha_{EM} / \alpha_W)^2$	$\bar{\delta}^{-2}$	0.053	0.066
$(\alpha_{EM} / \alpha_{QCD})^2$	$\bar{\delta}^{-4}$	4.034×10^{-3}	4.323×10^{-3}

Tab 2: Actual versus predicted ratios of SM parameters (except neutrinos)

Parameter ratio	Behavior	Actual	Predicted
m_ν^0 / m_e	$\bar{\delta}^{-8}$	$< 2\times10^{-7}$ $< 4\times10^{-6}$	1.87×10^{-5}
m_ν^0 / m_e	$\bar{\delta}^{-16}$	$< 2\times10^{-7}$ $< 4\times10^{-6}$	3.5×10^{-10}

Tab. 3: Actual vs. predicted ratios of neutrino mass scales.

4. SM AS A MULTIFRACTAL SET

In this section we argue that, *at least near the electroweak scale*, the SM represents a *self-contained multifractal set* on the MFM characterized by $D = 4 - \varepsilon,\ \varepsilon << 1$. All coupling charges residing on the MFM (gauge, Higgs and Yukawa) satisfy a closure relationship that a) tightly constrains the flavor and mass content of the SM and b) naturally solves the "hierarchy problem", without resorting to new concepts reaching beyond the physics of the SM.

This section is organized as follows: the relevant definitions and assumptions are introduced in paragraph 4.2; the modification of a generic action functional living on the MFM is detailed in 4.3. The next paragraph explores the consequences of placing classical electrodynamics of charged fermions on MFM. Expanding on these ideas, 4.5 reveals how the mass and flavor content of the SM may be derived from the properties of the MFM. The ensuing multifractal structure of the SM and the proposed resolution of the hierarchy problem form the topic of paragraphs 4.6 and 4.7.

4.1 DEFINITIONS AND ASSUMPTIONS

A4.1) As previously pointed out, the cross-over regime between $\varepsilon \neq 0$ and $\varepsilon = 0$ is the *only sensible setting* where the dynamics of interacting fields is likely to asymptotically approach all consistency requirements imposed by QFT and the SM. Large deviations from four dimensions ($\varepsilon \sim O(1)$) may signal the breakdown of these requirements. Particular attention needs to be paid, for example, to the potential violation of Lorentz invariance in Quantum Gravity theories advocating the emergence of space-time of lower dimensionality at high energy scales [1-3].

From the standpoint of interacting field theory, a non-vanishing and arbitrarily small deviation from four dimensions is equivalent to allowing the Renormalization Group (RG) equations to slide outside the isolated fixed points solutions (FP) [4]. Recalling that FP are synonymous with equilibria in the dynamical systems theory, it follows that, in general, the evolution of quantum fields is no longer required to settle down to equilibrium states. The end result is that the condition $\varepsilon << 1$ enables the isolated FP of the RG equations to morph into attractors with a more complex structure [4, 5].

A4.2) u_0 is the reference charge distribution on MFM for a *fixed* $\varepsilon << 1$ (fixed number of dimensions),

A4.3) $\overline{u}$ is the effective charge distribution on MFM when $\varepsilon << 1$ is allowed to vary (i.e., the number of dimensions is allowed to evolve with the energy scale),

A4.4) λ_0, g_0, $y_{0,f}$ are the coupling charges for the scalar, gauge and Yukawa sectors of the Standard Model, measured at the energy of the electroweak scale defined by M_{EW} in ordinary four dimensional space-time ($\varepsilon = 0$).

A4.5) Any theory exploring physics beyond the Standard Model (BSM) must fully recover the principles and the framework of perturbative QFT at energy scales approaching M_{EW}. In particular, it needs to preserve unitarity, renormalizability and local gauge invariance and be compatible with precision electroweak data [6, 7].

4.3 THE MINIMAL FRACTAL MANIFOLD (MFM)

Field theory on fractional four-dimensional space-time is described by the action

$$S = \int_{-\infty}^{+\infty} d\rho(x) L = \int_{-\infty}^{+\infty} (\mathrm{v}(x) d^4 x) L \qquad (4.1)$$

where the measure $d\rho(x)$ denotes the ordinary four-dimensional volume element multiplied by a weight function $\mathrm{v}(x)$ [2, 3]. If the weight function is factorizable in coordinates and positive semidefinite, $\mathrm{v}(x)$ assumes the form

$$\mathrm{v}(x) = \prod_{\eta=0}^{3} \frac{\left|x^\eta\right|^{\alpha_\eta - 1}}{\Gamma(\alpha_\eta)} \qquad (4.2)$$

in which

$$0 < \alpha_\eta \leq 1 \qquad (4.3)$$

are four independent parameters. An isotropic space-time of dimension $D = 4 \pm \varepsilon$ is characterized by

$$\alpha = 1 \pm \varepsilon = \frac{\sum_{\eta} \alpha_{\eta}}{4} \tag{4.4}$$

which turns (4.2) into

$$\mathrm{v}(x) \approx (|x|^{4})^{\pm \varepsilon} \tag{4.5}$$

Dimensional analysis requires all coordinates entering (4.2) and (4.5) to be scalar quantities. They can be generically specified relative to a characteristic length and time scale, as in

$$x = \frac{x_0}{L} = \frac{\mu}{\mu_0} \tag{4.6}$$

in which $\mu, \ \mu_0$ are positive-definite energy scales. Relation (4.5) becomes

$$\mathrm{v}(x) = (\frac{\mu}{\mu_0})^{\pm 4 \varepsilon} \tag{4.7}$$

such that

$$\lim_{|x| \to 0} \mathrm{v}(x) = \begin{cases} 0, \ if \ \pm \varepsilon > 0 \\ \infty, \ if \ \pm \varepsilon < 0 \end{cases} \tag{4.8}$$

Choosing $\mu < \mu_0$ we can expand (4.7) as:

$$a^{\varepsilon} = e^{\varepsilon \ln a} \approx 1 + \varepsilon \ln a \tag{4.9}$$

which yields

$$\mathrm{v}(x) = 1 \pm 4\varepsilon \ln(x) = 1 \pm 4\varepsilon \ln(\frac{\mu}{\mu_0}) \tag{4.10}$$

4.4 EMERGENCE OF EFFECTIVE FIELD CHARGES ON THE MFM

A remarkable property of fractal spacetime is the emergence of "effective" coupling charges induced by polarization in non-integer dimensions [8]. To fix ideas, consider the case of classical electrodynamics coupled to spinor fields in a MFM with evolving dimensionality [2]. From (4.10) we obtain

$$\bar{e}^2 = \mathrm{v}(x) e_0^2 \approx \frac{e_0^2}{1 \mp 4\varepsilon \ln(\frac{\mu}{\mu_0})} \tag{4.11}$$

where, following definitions A4.2) and A4.3),

$$\bar{e} = \bar{u}, \quad e_0 = u_0$$

In light of assumption A4.5), (4.11) has to match the expression of the running charge in perturbative Quantum Electrodynamics (QED). At one loop, this expression reads [9]

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2} \ln(\frac{\mu}{\mu_0})} \tag{4.12}$$

Comparing (4.11) with (4.12) leads to:

$$\boxed{e_0^2 = O(\varepsilon)} \tag{4.13}$$

This finding reveals that the dimensional parameter ε represents the physical source of the field charge in ordinary four-dimensional space-time. As previously alluded to, this "dynamic generation" of effective field charges can be traced back to the intrinsic polarization induced by fractal spacetime. The process is strikingly similar to the emergence of non-trivial FP's in the LGW model of critical behavior in $D=4-\varepsilon$ dimensions [4, 10]. The discussion may be extrapolated from electrodynamics to classical gauge theory and, as previously pointed out, it sets the stage for a novel interpretation of mass and flavor hierarchies present in the SM.

4.5 THE MASS AND FLAVOR HIERARCHIES OF THE SM

Re-iterating results obtained in section 3.3, the analysis of the RG equations on the MFM reveals that, near the electroweak scale, the normalized masses of fermions (m_f), weak bosons (M) and electroweak gauge charges (g_0) scale as

$$m_f \sim \varepsilon \tag{4.14}$$

$$g_0^2 \sim \varepsilon \tag{4.15}$$

$$g_0^2 M^2 = const \rightarrow M^2 \sim \varepsilon^{-1} \tag{4.16}$$

It can be also shown that, under some generic assumptions regarding the RG flow and its boundary conditions, the system of RG equations lead in general to a transition to chaos via period-doubling bifurcations as $\varepsilon \rightarrow 0$ [4, 11]. According to ideas outlined in section 3, the sequence of critical values ε_n, $n=1,2,...$ driving this transition to chaos satisfies the geometric progression

$$\varepsilon_n - \varepsilon_\infty = \varepsilon_n - 0 \sim k_n \overline{\delta}^{-n} \tag{4.17}$$

Here, $n >> 1$ is the index counting the number of cycles or tori created through the period-doubling cascade, $\overline{\delta}$ is the rate of convergence and k_n is a coefficient that becomes asymptotically independent of n as $n \to \infty$. Period-doubling cycles or tori are characterized by $n = 2^i$, for $i >> 1$. Substituting (4.17) in (4.14) and (4.15) yields the following ladder-like progression of critical couplings

$$\boxed{m_{f,i} \sim g_{0,i}^2 \sim \overline{\delta}^{-2^i}} \tag{4.18a}$$

In section 3.3 we found that scaling (4.18a) recovers the full mass and flavor content of the SM, including neutrinos, together with the coupling strengths of gauge interactions. Specifically,

- The trivial FP of the RG equations consists of the massless photon (γ) and the massless UV gluon (g).
- The non-trivial FP of the RG equations is degenerate and consists of massive quarks (q), massive charged leptons and their neutrinos (l, ν) and massive weak bosons (W, Z).
- Gauge interactions develop near the non-trivial FP and include electrodynamics, the weak interaction and the strong interaction.

It was suggested in [4] that a space-time background with low-level fractality ($\varepsilon << 1$) favors the formation of a *Higgs-like condensate* of gauge bosons, as in

$$\Phi_C = \tfrac{1}{4}[(W^+ + W^- + Z^0 + \gamma + g) + (W^+ + W^- + Z^0 + \gamma + g)] \qquad (4.18b)$$

Here, $W^{\pm}, Z^0$ denote the triplet of electroweak bosons and g, γ stand for gluon and photon, respectively. Relation (4.18b) implies that the scalar condensate Φ_C acquires a mass in agreement with the mass of the SM Higgs ($m_H = 125.6\,GeV$). In closing, we note that tantalizing hints on a "would be" heavy charged boson W' or W_R formed from di-boson condensates (WW, WZ, ZZ) have been reported in [13, 14].

4.6 MULTIFRACTAL STRUCTURE OF THE SM

A key parameter of the RG analysis is the dimensionless ratio (μ/Λ_{UV}), in which μ is the sliding scale and $\Lambda_{UV} >> \mu$ the high-energy cutoff of the underlying theory. As discussed in the first section, the connection between the parameter $\varepsilon = 4 - D$ and Λ_{UV} is given by

$$\varepsilon \sim \frac{1}{\log(\frac{\Lambda_{UV}^2}{\mu^2})} \qquad (4.19)$$

The large numerical disparity between μ and Λ_{UV} enables one to approximate ε as in

$$\varepsilon \sim (\frac{\mu}{\Lambda_{UV}})^2 \qquad (4.20)$$

Let m_i denote the full spectrum of particle masses present in the SM. Relation (4.20) can be written as

$$\varepsilon_i = (\frac{m_i}{\Lambda_{UV}})^2 = \frac{m_i^2}{M_{EW}^2}\frac{M_{EW}^2}{\Lambda_{UV}^2} = r_i^2 \varepsilon_0 \qquad (4.21)$$

in which

$$r_i = \frac{m_i}{M_{EW}}, \quad \varepsilon_0 = \frac{M_{EW}^2}{\Lambda_{UV}^2} \tag{4.22}$$

and

$$r_i^2 = \frac{\varepsilon_i}{\varepsilon_0} \tag{4.23}$$

With reference to (c.3) of Appendix C, we find that (4.23) obeys a closure relationship typically associated with multifractal sets, namely:

$$\boxed{\sum_i r_i^2 = \sum_i \left(\frac{m_i}{M_{EW}}\right)^2 = 1} \tag{4.24}$$

in which the sum in the left-hand side extends over all SM fermions (leptons and quarks).

The sum rule (4.24) may be alternatively cast in terms of SM field charges. We obtain

$$2\lambda_0 + \frac{g_0^2}{4} + \frac{g_0^2 + (g_0')^2}{4} + \sum_{l,q} \frac{y_{0,f}^2}{2} = 1 \tag{4.25}$$

where

$$\lambda_0 = \frac{(u_0)_{scalar}}{\varepsilon_0}$$

$$g_0^2 = \frac{(u_0)_{gauge}}{\varepsilon_0}$$

$$g'^2_0 = \frac{(u'_0)_{gauge}}{\varepsilon_0}$$

From either (4.24) or (4.25) one derives

$$\boxed{M_{EW} \sim V = 246.2\ GeV} \tag{4.26}$$

in close agreement with the vacuum expectation value of the SM Higgs boson (V). It is instructive to note that the existence of (4.25) was first brought up in [12], with no attempt of formulating a theoretical interpretation.

We close this paragraph with another intriguing relationship involving the set of multifractal scales (4.24). Sections 7 and Appendix E below, as well as to [15, 16], point to a subtle link between the onset of MFM and the quantum spin. In particular,

- The MFM appears to provide a topological interpretation of quantum spin as direct outcome of $\varepsilon << 1$ near M_{EW}.
- The approach to scale invariance near $\varepsilon \to 0$ forces the action functional to be independent of any particular choice of ε.

Taken together, these observations hint that the onset of the MFM leads to a *topological symmetry* between bosons and fermions that is absent on the conventional four-dimensional spacetime. This symmetry evades the constraints imposed by the Coleman-Mandula theorem [17] and suggests the equipartition of fermionic and bosonic scales on the MFM, that is,

$$\sum_i (r_i^2)_{fermions} = \sum_j (r_j^2)_{bosons} \tag{4.27}$$

Taking into account (4.24) yields

$$\boxed{m_{top}^2 + m_{bottom}^2 + m_{lept}^2 + m_{light\ quarks}^2 \approx M_W^2 + M_Z^2 + m_H^2} \qquad (4.28)$$

which is found to hold to within 1% numerical accuracy.

4.7 SOLVING THE FLAVOR AND HIERARCHY PROBLEMS ON THE MFM

Relations (4.18), (4.24) and (4.25), (4.28) tightly constrain the particle content of the SM. They naturally fix its number of independent field flavors near the electroweak scale. Also, since all scaling ratios in (4.24) must have a magnitude of less than one unit, (4.24) and (4.25) necessarily imply that the mass of the Higgs boson cannot grow beyond M_{EW}, at least near the electroweak scale. This conclusion brings closure to the *hierarchy problem*, whose formulation is briefly outlined in Appendix A.

5. MFM AND THE DYNAMIC GENERATION OF MASS SCALES IN FIELD THEORY

The consensus among high-energy theorists is that, as of today, the mechanism underlying the generation of mass scales in field theory remains elusive. Our intent here is to point out that the MFM can naturally account for the onset of these scales. A counterintuitive outcome of this analysis is the deep link between the minimal fractal manifold and the holographic principle.

5.1 MOTIVATION

One of the many unsettled questions raised by field theory revolves around the vast hierarchy of scales in Nature [1-3]. A large numerical disparity exists between the Planck scale (M_{Pl}), the electroweak scale (M_{EW}), the hadronization scale of Quantum Chromodynamics (Λ_{QCD}) and the cosmological constant scale ($\Lambda_{cc}^{1/4}$, with Λ_{cc} expressed as energy density in 3+1 dimensions).

It has been long known that perturbative QFT cannot provide a complete description of Nature since its formalism entails divergences at both ends of the energy spectrum [1, 2 , 4]. For instance, many textbooks emphasize that the singular behavior of momentum integrals in the ultraviolet (UV) sector arises from the poorly understood space-time structure at short distances [1, 2]. Lattice field models handle infinities through discretization of the space-time continuum on a grid of spacing "Δ". This procedure naturally bounds the maximal momentum allowed to propagate through the lattice, namely,

$$p \leq p_{\max} \sim (2\Delta)^{-1} \tag{5.1}$$

The downside of lattice models is that they generally fail to be either gauge or Poincaré invariant [1, 2, 4, 5]. Restoring formal consistency is further enabled via the RG program [1, 2, 6]. RG regulates the *n*-th order momentum integrals of the generic form

$$I_n(p) = \int dp\, f(p^{2n}) \tag{5.2}$$

by either inserting an arbitrary momentum cutoff $0 < \Lambda \sim \Delta^{-1} < \infty$ *or* by continuously "deforming" the four-dimensional spacetime via the dimensional parameter ε . The resulting theory is free from divergences and operates with a finite number of redefined physical parameters. Restoring the continuum spacetime limit is done at the end by taking the limit $\Lambda \to \infty$ or $\varepsilon \to 0$. Both limits are disfavored by experimental data, as discussed in section 3.

Reinforcing this viewpoint, some authors argue that the idea of smooth space-time stands in manifest conflict with the basic premises of quantum theory [7]. To confine an event within a region of extension Δ requires a momentum transfer on the order of Δ^{-1} which, in turn, generates a local gravitational field. If the density of momentum transfer is comparable in magnitude with the right hand side of Einstein's equation, the local curvature of space-time ($\sim R_0^{-2}$) induced by this transfer is given by (in natural units, $\hbar = c = 1$)

$$R_0^{-2} \sim G_N \Delta^{-4} \tag{5.3}$$

However, collapse of the event within a short region of extent $\Delta = O(R_0)$ amounts to trapping outgoing light signals and preventing direct observation.

All these considerations invariably point to the following challenge: on the one hand, a continuum model of space-time near or below M_{EW} serves as an effective paradigm that is likely to fail at large probing energies. Yet on the other, any discrete model of space-time typically violates Poincaré or gauge symmetries. It seems only natural, in this context, to take a fresh look at (1.8), (4.19) and (4.20) and appreciate the message it conveys: if either Λ_{UV} stays finite *or* ε << 1 is arbitrarily small but non-vanishing, space-time dimensionality becomes a non-integer arbitrarily close to four. Stated differently, in the neighborhood of M_{EW} , conventional space-time necessarily turns into a MFM [6, 8-11].

On closer examination, this finding is hinted by a number of alternative theoretical arguments:

a) It is well known that the principle of *general covariance* lies at the core of classical relativistic field theory. An implicit assumption of general covariance is that any coordinate transformation and its inverse are *smooth* functions that can be differentiated arbitrarily many times. However, as it is also known, there is a plethora of non-differentiable curves and surfaces in Nature, as repeatedly discovered since the introduction of fractal geometry in 1983 [12, 13]. The unavoidable conclusion is that relativistic field theory assigns a preferential status to differentiable transformations and the smooth geometry of spacetime, which is at odds with the very spirit of general covariance.

b) On the mathematical front, significant effort was recently invested in the development of q-deformed Lie algebras, non-commutative field theory, quantum groups, fractional field theory and its relationship to the MFM [4, 14-18]. It is instructive to note that all these contributions appear to be directly or indirectly related to fractal geometry [19]. Moreover, as repeatedly stated, the condition $\varepsilon << 1$, defined within the framework of MFM, is the sole sensible setting where fractal geometry asymptotically approaches all consistency requirements mandated by QFT and the Standard Model.

c) Demanding that phenomena associated with gravitational collapse follow the postulates of quantum theory implies that the world is no longer four-dimensional near M_{Pl}. This statement has lately received considerable attention and forms the basis for *dimensional reduction* and for the *holographic principle* of Quantum Gravity (QG) theories [3, 17-25]. If we accept that the four-dimensional continuum is an emergent property of the electroweak scale and below ($\mu < M_{EW}$), the holographic principle implies that space-time dimensionality evolves with the energy scale between M_{EW}, where ε << 1, and M_{Pl}, where space is expected to become two-dimensional viz. $\varepsilon = O(1)$.

The section is organized as follows: next paragraph introduces the concept of holographic bound and derives the relationship involving the IR and UV cutoffs of field theory. Building on these premises, 5.3 develops a comparison between mass scales estimated using our approach and their currently known values.

Before proceeding, we bring up an important observation. Following [19], the far infrared (IR) scale of field theory set by the cosmological constant ($\Lambda_{cc}^{1/4}$), the electroweak scale (M_{EW}) and the far ultraviolet (UV) scale fixed by the Planck mass scale (M_{Pl}) satisfy the constraint

$$\boxed{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \sim \frac{M_{EW}}{M_{Pl}} = O(\varepsilon)} \tag{5.4}$$

A direct outcome of the MFM philosophy, relation (5.4) mixes and constrains largely separated scales and may play a key role in explaining the hierarchy problems of field theory.

5.2 THE HOLOGRAPHIC BOUND

Consider an effective QFT confined to a spacetime region with characteristic length scale L and assume that the theory makes valid predictions up to an UV cutoff scale Λ_{UV} >> L^{-1}. It can be shown that the entropy associated with this effective QFT takes the form [20]

$$S \sim \Lambda_{UV}^3 L^3 \tag{5.5}$$

To understand the significance of (5.5), consider an ensemble of fermions living on a periodic space lattice with characteristic size L and period Λ_{UV}^{-1}. One finds that (5.5) simply follows from counting the number of occupied states for this system, which turns out to be $N = 2^{(L\Lambda_{UV})^3}$ [20]. The holographic principle stipulates that (5.5) must not exceed the corresponding black hole entropy S_{BH}, that is,

$$L^3\Lambda_{UV}^3 \leq S_{BH} = \frac{A_{BH}}{4l_{Pl}^2} = \pi R^2 M_{Pl}^2 \tag{5.6a}$$

in which A_{BH} is the area of the spherical event horizon of radius R. Introducing a new reference length scale Δ defined as

$$\Delta = \frac{L^3}{R^2} \tag{5.6b}$$

leads to the condition

$$\Delta \leq \pi \Lambda_{UV}^{-3} M_{Pl}^{2} \tag{5.7}$$

On the other hand, since the maximum energy density in a QFT bounded by the UV cutoff is Λ_{UV}^4, the holography bound (5.6a) leads to [26, 27]

$$\Lambda_{UV}^{4} \sim \frac{(\pi^{-1}\Delta) M_{Pl}^{2}}{(\pi^{-1}\Delta)^{3}} = \pi^{2} \frac{M_{Pl}^{2}}{\Delta^{2}} \Rightarrow \Lambda_{UV}^{2} \sim \pi \frac{M_{Pl}}{\Delta} \tag{5.8}$$

Since the IR cutoff is fixed by $\Lambda_{IR} = \Delta^{-1}$, (5.8) yields the scaling behavior

$$\boxed{\frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi M_{Pl}}} \tag{5.9}$$

Although conventional wisdom suggests that the SM retains its validity all the way up in the far UV sector of particle physics, there are indications that it may break at a scale that is at least an order of magnitude lower than M_{Pl}, that is, $\Lambda'_{UV} < M_{Pl}$ [28]. Relation (5.9 may be conveniently reformulated at $\Lambda'_{UV} > \Lambda_{UV}$ as in

$$\frac{\Lambda_{UV}}{\pi M_{Pl}} = \frac{\Lambda_{UV}}{\pi \Lambda'_{UV}} \frac{\Lambda'_{UV}}{M_{Pl}} \tag{5.10}$$

such that

$$\frac{M_{Pl}}{\Lambda'_{UV}} \frac{\Lambda_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi \Lambda'_{UV}} \tag{5.11}$$

or

$$\boxed{\frac{\Lambda'_{IR}}{\Lambda_{UV}} \sim \frac{\Lambda_{UV}}{\pi\Lambda'_{UV}}} \quad (5.12)$$

in which $\Lambda'_{IR} > \Lambda_{IR}$ is a new IR scale given by

$$\Lambda'_{IR} = \frac{M_{Pl}\,\Lambda_{IR}}{\Lambda'_{UV}} \quad (5.13)$$

A glance at (5.4), (5.9) and (5.12) reveals deep similarities between the holographic principle and the MFM. They all represent scaling relations that mix and constrain largely separated mass scales. We next use (5.9) and (5.12) to derive numerical estimates and compare them with experimental data.

5.3 NUMERICAL ESTIMATES

Tab. 4 displays currently known values for the representative scales of QFT and classical field theory. The electroweak scale (M_{EW}) is set by the vacuum expectation value of the Higgs boson, the far UV scale is set by either Planck mass (M_{Pl}) or the postulated unification scale (M_{GUT}). The near UV cutoff is assumed to be close to the so-called Cohen-Kaplan threshold ($\Lambda_{CK} \sim 10^2$ TeV), according to [26, 28-31].

Scale	Name	Magnitude
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	Cosmological constant scale	$\leq \sim 10^{-3}$ eV
$\Lambda'_{IR} = \Lambda_{QCD}$	QCD scale	~ 200 MeV
$\Lambda_{UV} = M_{EW}$	EW scale	~ 246 GeV
$\Lambda'_{UV} = \Lambda_{CK}$	UV cutoff	~ 10^2 TeV
M_{GUT}	GUT scale	~ 10^{16} GeV
M_{Pl}	Planck scale	~ 10^{19} GeV

Tab. 4: The spectrum of mass scales in field theory

Tab. 5 shows numerical results. We find that:

a) the cosmological constant scale is consistent with its experimentally determined value and with the scale of neutrino masses [32].

b) the near IR scale is consistent with the QCD scale (Λ_{QCD}). This conclusion may shed light into the long-standing problem of the QCD mass gap as well as on the non-perturbative properties of strongly coupled gauge theory [27, 33, 34].

Mass scale	Estimated	Units	Comments
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.6\times10^{-6}$	eV	from M_{Pl}
$\Lambda_{IR} = \Lambda_{cc}^{1/4}$	$\sim 1.9\times10^{-3}$	eV	from M_{GUT}
$\Lambda'_{IR} = \Lambda_{QCD}$	~ 193	MeV	from Λ_{CK}

Tab 5: Estimated values of the cosmological constant and QCD scales (assuming the electroweak scale at $M_{EW} \approx$ 246 GeV and the Cohen-Kaplan cutoff at $\Lambda_{CK} \approx 10^2$ TeV)

The hierarchy of mass scales derived above can be conveniently summarized in the following diagram:

$$\Lambda_{cc}^{1/4} \text{ (far IR Cutoff)} << \Lambda_{QCD} \text{ (near IR cutoff)} < M_{EW} < \Lambda_{CK} \text{ (near UV cutoff)} << M_{Pl} \text{ (far UV cutoff)}$$

6. CHARGE QUANTIZATION ON THE MFM

This section briefly makes the case that classical Maxwell equations on fractal distributions can account for the quantization of electric charge. In contrast with the standard formulation of classical electrodynamics, Maxwell equations on fractal distribution of charged particles generate *fractional magnetic charges* or *fractional* *monopoles* (q_m) [1, 2]. Although these fractional objects are un-observable at energy scales significantly lower than M_{EW}, their cumulative contribution may become relevant for charge quantization following Dirac's theory of magnetic monopoles. Needless to say, this short analysis is far from being either rigorous or complete. Our sole intent here is opening an unexplored research avenue which, to the best of our knowledge, has not received any prior consideration.

The non-vanishing divergence of an external magnetic field $\mathbf{B}$ applied to a fractal distribution of charges is given by

$$\nabla \cdot \mathbf{B} = -\mathbf{B} \cdot \nabla c_2(d, \mathbf{r}) \tag{6.1}$$

in which the correction coefficient assumes the form

$$c_2(d, \mathbf{r}) = \frac{2^{2-d}}{\Gamma(d/2)} |\mathbf{r}|^{d-2} \tag{6.2}$$

Fractional monopoles depend on the gradient of (6.2) according to

$$q_m \sim \mathbf{B} \cdot \nabla c_2(d, \mathbf{r}) \tag{6.3}$$

We assume herein that the magnitude of the radial vector $\mathbf{r}$ is normalized to a reference length r_0 or, equivalently, to a reference mass scale $\mu_0 = r_0^{-1}$. Hence,

$$\mathbf{r} = (\frac{r}{r_0})\mathbf{u}_\mathbf{r} = (\frac{\mu_0}{\mu})\mathbf{u}_\mathbf{r} \tag{6.4}$$

in which $\mathbf{u_r}$ stands for the unit vector in the radial direction. Since the deviation from two dimensionality on a minimal fractal manifold is quantified as $d = 2 \pm \varepsilon$, with ε << 1, (6.2) is well approximated by

$$c_2(d,\mathbf{r}) \sim (\frac{\mu_0}{\mu})^{\pm\varepsilon} \tag{6.5}$$

Combined use of (6.2) and (6.5) yields

$$\nabla c_2(\varepsilon,\mathbf{r}) \sim \pm\varepsilon(\frac{\mu_0}{\mu})^{-1}\mathbf{u_r} = \pm\varepsilon(\frac{\mu}{\mu_0})\mathbf{u_r} \tag{6.6}$$

Because our analysis is carried out in a classical framework, we choose $\mu_0 = M_{EW}$ and the regime of mesoscopic scales $\mu << M_{EW}$, with $\mu / M_{EW} = O(\varepsilon)$. Relation (6.6) turns into

$$\nabla c_2(\varepsilon,\mathbf{r}) \sim \pm\varepsilon^2\mathbf{u_r} \tag{6.7}$$

The quadratic dependence on ε suggests that fractional magnetic charges are likely to be unobservable on mesoscopic scales. Substituting (6.7) into the Dirac charge quantization condition [3] gives

$$eq_m \sim \frac{n}{2} \Rightarrow e(\pm\varepsilon^2\mathbf{B}\cdot\mathbf{u_r}) \sim \frac{n}{2} \tag{6.8}$$

where natural units are assumed and $n = \pm 1, \pm 2, \ldots$. It is readily seen that, in contrast with fractional magnetic charges, the quantization of free electric charges scales as ε^{-2} and is likely to be observable at mesoscopic distances on the order of $O(\mu^{-1})$.

A key point regarding (6.8) is now in order. The standard Dirac's theory of magnetic monopoles is based on the concept of "*point charges*". Seen in the context of quantum electrodynamics (QED) and its renormalization procedure, point charges are singular and correspond to "bare" quantities indexed by the subscript "*B*" [4, 5]. Following this interpretation, Dirac's quantization condition may be re-written as

$$e_B (q_m)_B \sim \frac{n}{2} \tag{6.9}$$

The asymptotic behavior of QED at 1-loop computations is given by

$$e_B \sim e(1+\frac{e^2}{12\pi^2\varepsilon}) = eO(\varepsilon^{-1}) \tag{6.10}$$

where "e"stands for the renormalized and finite electric charge.

A natural assumption is that magnetic charges obey a similar behavior, namely

$$(q_m)_B \sim q_m(1+\frac{q_m}{12\pi^2\varepsilon}) = q_m O(\varepsilon^{-1}) \tag{6.11}$$

Using (6.10) and (6.11) turns (6.9) into

$$e q_m O(\varepsilon^{-2}) \sim \frac{n}{2} \tag{6.12}$$

According to (6.7) and (6.8), the "effective" magnetic charge is nearly vanishing in the infrared limit of mesoscopic scales. Thus,

$$q_m \sim (\pm\varepsilon^2 \mathbf{B}\cdot\mathbf{u_r}) = O(\varepsilon^2) \tag{6.13}$$

Substituting (6.13) in (6.12) leads to a quantization condition that is independent of ε and expressed exclusively in terms of *finite quantities*.

7. ON THE CONNECTION BETWEEN THE MFM AND QUANTUM SPIN

The aim of this section is to point out that the inner connection between MFM and local conformal field theory (CFT) makes *quantum spin* a topological property of the MFM.

7.1 INTRODUCTORY REMARKS

In his seminal paper of 1939, Wigner has shown that the concept of *quantum spin* follows naturally from the unitary representation of the Poincaré group [1-3]. The two invariant Casimir operators of the Poincaré group, $P_\mu P^\mu = m^2$ and $W^\mu W_\mu = -ms(s+1)$ supply the rest mass m and the spin s of the particle, respectively. Here P^μ is the generator of translations and W^μ the Pauli-Lubanski operator defined as

$$W^\mu = \varepsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} \tag{7.1}$$

in which $\varepsilon^{\mu\nu\rho\sigma}$ stands for the four-dimensional Levi-Civita index and $J^{\mu\nu}$ are the generators of the Lorentz group. The second Casimir invariant implies that the square of the spin three-vector of a massive particle ($\mathbf{S}$) relates to the Pauli-Lubanski operator via

$$\mathbf{S} \cdot \mathbf{S} = \frac{1}{m^2} W^\mu W_\mu \tag{7.2}$$

Our brief analysis reveals that quantum spin may be understood outside the traditional framework of representation theory, specifically as emerging attribute of the MFM. Expanding on these ideas, we next suggest that the inner connection between MFM and local conformal field theory (CFT) makes quantum spin a topological property of the MFM. It is instructive to note that this interpretation of quantum spin resonates well with the framework of ideas presented in [4].

7.2 QUANTUM SPIN AS MANIFESTATION OF THE MFM

Consider a flat four-dimensional spacetime with constant metric having the standard signature $\eta_{\mu\nu} = diag(-1,...,+1)$. A differentiable map $x' = \zeta(x)$ is called a *conformal transformation* if the metric tensor changes as [5]

$$\eta_{\mu\nu} \rightarrow \overline{\eta_{\mu\nu}} = \eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Omega^2(x)\eta_{\mu\nu} \qquad (7.3)$$

in which $\Omega^2(x)$ represents the scale factor and Einstein's summation convention is implied. The scale factor is *strictly equal to unity* on flat space-times ($\Omega^2(x)=1$), a condition matching the translations and rotations group of Lorentz transformations. In general, if the underlying spacetime background deviates from flatness and is characterized by a metric $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$, the condition for local conformal transformation (7.3) reads

$$g_{\mu\nu}(x) \rightarrow \overline{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \qquad (7.4)$$

where $\Omega^2(x) \neq 1$. A *nearly conformal transformation* (NCT) is defined by a scale factor departing slightly and continuously from unity, that is,

$$\Omega^2(x) = 1 + \varepsilon(x) \approx \exp[\varepsilon(x)] \;, \;\; \varepsilon(x) << 1 \qquad (7.5)$$

Consider next infinitesimal coordinate transformations which, up to a first order in a small parameter $\upsilon(x) << 1$, can be presented as

$$x'^{\rho} = x^{\rho} + \upsilon^{\rho}(x) + O(\upsilon^{2}) \tag{7.6}$$

Demanding that (7.6) represents a local conformal transformation amounts to [5]

$$\partial_{\mu}\upsilon_{\nu} + \partial_{\nu}\upsilon_{\mu} = \frac{2}{D}(\partial\cdot\upsilon)\,\eta_{\mu\nu} \tag{7.7}$$

The scale factor corresponding to (7.6) is given by

$$\Omega^{2}(x) = 1 + \frac{2(\partial\cdot\upsilon)}{D} + O(\upsilon^{2}) \tag{7.8}$$

Any locally defined MFM is characterized by a spacetime dimension $D(x) = 4 - \varepsilon(x)$, where the onset of the fractal dimension $\varepsilon(x) << 1$ reflects a nearly-vanishing deviation from strict conformal invariance expected at the trivial fixed points FP's of the RG flow [6-8]. Conformal behavior in flat space-time matches the scale-invariant (constant) metric $\eta_{\mu\nu}$, whereby $\Omega^{2}(x) \to 1$ and $\varepsilon(x) \to 0$ as a result of (7.3) and (7.5). In field-theoretic language, reaching the conformal limit on the flat four dimensional spacetime means that the typical RG trajectories flow into trivial FP's where they settle down to steady equilibria. One arrives at similar conclusions by following the prescription of the dimensional regularization program [6-8]. All these observations enable us to draw a natural connection between the fractal dimension $\varepsilon(x) << 1$ and the NCT, namely,

$$D(x) = 4 - \varepsilon(x) \Leftrightarrow \Omega^{2}(x) = 1 + \varepsilon(x) \tag{7.9}$$

Replacing (7.9) into (7.8) and ignoring the contribution of quadratic terms yields

$$2\varepsilon(x) - \partial \cdot \upsilon(x) \ll 1 \qquad (7.10)$$

Furthermore, setting the fractal dimension as divergence of a locally defined "dimensional" field $\xi(x)$

$$2\varepsilon(x) = \partial_\mu \xi^\mu = \partial \cdot \xi \qquad (7.11)$$

leads to the following condition for conformal invariance on the MFM

$$\partial \cdot (\upsilon - \xi) \ll 1 \qquad (7.12)$$

A typical ansatz in CFT is to assume that the infinitesimal coordinate transformations $\upsilon_\mu(x)$ are at most quadratic in x^ν, that is,

$$\upsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \qquad (7.13)$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ are constant coefficients with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. The individual terms of expansion (13) describe various conformal transformations and their respective generators. In particular,

1) The constant coefficient a_μ represents an infinitesimal translation $x'^\mu = x^\mu + a^\mu$ whose generator is the momentum operator $P_\mu = -i\partial_\mu$.

2) The next term can be split into a symmetric and an anti-symmetric contribution according to

$$b_{\mu\nu} = \lambda\eta_{\mu\nu} + m_{\mu\nu} \tag{7.14}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. The symmetric part $\lambda\eta_{\mu\nu}$ labels infinitesimal scale transformations (dilatations) of the generic form $x'^{\mu} = (1+\lambda)x^{\mu}$ and corresponding generator $D = -ix^{\mu}\partial_{\mu}$. The anti-symmetric part $m_{\mu\nu}$ describes infinitesimal rotations $x'^{\mu} = (\delta^{\mu}_{\nu} + m^{\mu}_{\nu})x^{\nu}$ whose associated generator is the angular momentum operator $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - \partial_{\mu}x_{\nu})$.

3) The last term at the quadratic order in x defines the so-called "special conformal transformations".

Returning to (7.9) to (7.12), a reasonable hypothesis is to assume that the dimensional field $\xi(x)$ is at most linear in x, which corresponds to a nearly-constant fractal dimension $\varepsilon(x) \approx \varepsilon$. Thus we take

$$\xi_{\mu}(x) = d_{\mu} + e_{\mu\nu}x^{\nu} \tag{7.15}$$

subject to the requirement of infinitesimal coefficients $d_{\mu}, e_{\mu\nu} << 1$. Retracing previous steps, we split $e_{\mu\nu}$ into a symmetric and anti-symmetric contribution

$$e_{\mu\nu} = \lambda\eta_{\mu\nu} + f_{\mu\nu} \tag{7.16}$$

subject to the condition $f_{\mu\nu} = -f_{\nu\mu}$. The symmetric part denotes a scale transformation similar to $x'^{\mu} = (1+\lambda)x^{\mu}$, whereas the anti-symmetric part defines an "intrinsic" rotation of the form

$$\boxed{x'^{\mu} = (\delta^{\mu}_{\nu} + f^{\mu}_{\nu})\, x^{\nu}} \tag{7.17}$$

It follows that the "rotation-like" transformation (17) stems from the fractal topology of the MFM and may be associated with the generator of *quantum spin* $S_{\mu\nu}$. A favorable consequence of this brief analysis is that, by construction, $S_{\mu\nu}$ replicates the algebra of the angular momentum operator $L_{\mu\nu}$. In closing, we mention that these findings are consistent with the body of ideas developed in [6].

8. MINIMAL FRACTAL MANIFOLD AS ASYMPTOTIC REGIME OF NON-COMMUTATIVE FIELD THEORY

In this section we argue that MFM may be treated as asymptotic manifestation of Non-Commutative (NC) Field Theory near the electroweak scale. Our provisional findings may be further expanded to bridge the gap between MFM and NC Field Theory.

8.1 INTRODUCTION

Non-Commutative field theory represents a generalization of standard QFT to spacetimes having non-commuting coordinates. It is based on the premise that coordinates may be promoted to hermitean operators x^{μ} ($\mu = 0,1,2,3$) obeying the commutation rules [1, 3-5]

$$\left[x^{\mu}, x^{\nu} \right] = i\,\theta^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ is a real-valued and anti-symmetric matrix of dimension (length)². If $\theta^{\mu\nu}$ is constant, the commutators define a *Heisenberg algebra* and imply the space-time uncertainty

$$\Delta x^{\mu} \Delta x^{\nu} > \frac{1}{2} \left| \theta^{\mu\nu} \right| \tag{2}$$

It is known that space-time quantization (1) involves a number of difficulties when gauged against the geometry of four-dimensional continuum. For example, the condition $\theta^{0i} \neq 0$, $i = 1,2,3$ implies a theory that violates *causality* and *unitarity*. Likewise, (1) stands in conflict with *Lorentz invariance*: the choice $\theta^{12} \neq 0$ leads to breaking of Lorentz invariance to the residual $SO(1,1) \times SO(2)$ symmetry generated by boosts along the third space direction (3) and rotations in the $(1,2)$ directions [6].

As with any compelling efforts aimed at developing QFT beyond its present boundaries, NC field theory must be able to recover the physics of the Standard Model in the appropriate limit. In particular it has to fulfill all consistency requirements mandated by the Standard Model near the electroweak scale. It is our opinion that NC field theory, despite advancing many attractive claims, is not yet at this stage. As explained in the text, there are reasons to believe that the only way NC field theory can make sensible contact with the physics of the Standard Model is to conjecture that (1) can be mapped to a *continuous deformation* of conventional commutation rules. Moreover, this deformation must be dependent on a parameter that vanishes identically on the four-dimensional space-time. The goal of this section is to point out that the concept of MFM provides a natural choice for this conjecture.

A counterintuitive outcome of field theory is that the exact continuum limit of a local QFT formulated on flat spacetime has, strictly speaking, *no correlate to physical reality* [7]. The Minkowski metric of Special Relativity underlies the most basic aspect of QFT, namely the space-like commutativity of local observables, yet is considered only an "emergent" phenomenon and an approximate description of an underlying fundamental theory.

8.2 NON-COMMUTATIVITY OF FRACTAL OPERATORS

In a nut-shell, fractal (or fractional) operators are differential derivatives and integrals of arbitrary non-integer order. They offer novel tools for the analysis of interacting systems that are embedded on fractal supports or in dynamic environments falling outside equilibrium conditions. We survey next the commutativity of fractal operators with emphasis on the setting describing minimal fractality ($\varepsilon << 1$). Let

$$n-1 < \alpha < n,\ m-1 < \beta < m \text{ where } n,m \in N^{+} \tag{3}$$

denote the fractional order for two Caputo operators O^{α}, O^{β} working on a generic function $f(x)$ [2]. Their commutator is given by

$$\left[O^{\alpha}, O^{\beta}\right] = O^{\alpha}O^{\beta} - O^{\beta}O^{\alpha} \tag{4a}$$

We introduce the convention

$$O^{\alpha} = \partial^{\alpha},\ \text{if } \alpha > 0 \tag{4b}$$

$$O^{\alpha} = I^{\alpha},\ \text{if } \alpha < 0 \tag{4c}$$

To model the behavior of (4b-c) on the MFM and establish connection to the NC field theory, we take

$$\alpha = \varepsilon << 1, \ \ \varepsilon = 4 - D > 0 \tag{5a}$$

$$\beta = \varepsilon' << 1, \ \ \varepsilon' = 4 - D < 0 \tag{5b}$$

$$f(x^{\mu}) = x^{\mu} \tag{5c}$$

(5c) asymptotically converges to the conventional spacetime coordinates in the limit $\varepsilon, \varepsilon' \to 0$, that is,

$$\lim_{\varepsilon \to 0} (\partial^{\varepsilon})(x^{\mu}) = x^{\mu} \tag{6a}$$

$$\lim_{\varepsilon' \to 0} (I^{-\varepsilon'})(x^{\nu}) = x^{\nu} \tag{6b}$$

Using calculations detailed in [2] yields

$$\left[\partial^{\varepsilon}, I^{-\varepsilon'}\right] f(x) = 2 \sum_{j=0}^{n-1} \frac{(x - x_0)^{j - \varepsilon' - \varepsilon}}{\Gamma(j + 1 - \varepsilon' - \varepsilon)} (\partial^{j} f)(x_0) \tag{7}$$

where the space-time index μ, ν is omitted for the sake of clarity. The commutator vanishes if

$$(\partial^{j} f)(x_0) = 0, \quad j = 0, 1, 2 \ldots., n - 1. \tag{8}$$

which fails to be true unless $x_0 = 0$. Same conclusion applies to the case where the two operators are of the Riemann-Liouville type [2]. It is readily seen from (6a-b) and (7),

(8) that fractal operators working on the MFM enable a *continuous deformation* of space-time commutativity into the quantization condition (1). The deformation goes away as $\varepsilon, \varepsilon' \to 0$, a setting that recovers the familiar geometry of the four-dimensional continuum.

9. FRACTAL PROPAGATORS AND THE ASYMPTOTIC SECTORS OF QFT

This section contemplates the connection between the asymptotic regions of QFT and the MFM. The starting point of our analysis is the observation that propagators for charged fermions no longer follow the prescription of perturbative QFT in the far IR and far UV sectors of particle physics. The propagators acquire a fractal structure from radiative corrections contributed by gauge bosons. We show how this structure may be analyzed using the attributes of the MFM. An intriguing consequence of this approach is the emergence of classical gravity as long-range and ultra-weak excitation of the Higgs condensate.

9.1 INTRODUCTORY REMARKS

The free-fermion propagator in QFT determines the probability amplitude for a fermion to travel between different spacetime locations. It is given by [1-2]

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-ip\cdot(x-y)] S_F(p) \tag{9.1}$$

in which

$$S_F(p) = \frac{\gamma^\mu p_\mu + m}{p^2 - m^2 + i0^+} = \frac{1}{\gamma^\mu p_\mu - m + i0^+} \tag{9.2}$$

This formula successfully applies to both the IR regime of quantum electrodynamics (QED) and the UV limit of quantum chromodynamics (QCD), where the approximation of nearly free-fermions holds well. In contrast, at distance scales where the radiative contribution of soft photons to electron self-interaction becomes relevant and is accounted for, the propagator changes to [3-4]

$$S(p) = (\frac{m}{i\Lambda})^{\gamma}\Gamma(1+\gamma)\frac{\gamma^{\mu}p_{\mu} + m}{(p^2 - m^2 + i0^+)^{(1+\gamma)}} \tag{9.3}$$

Here, the fractional "anomalous" exponent $\gamma = \frac{\alpha}{\pi}$ is related to the low-energy value of the fine structure constant α, Λ is an arbitrary high-energy scale and $\Gamma(...)$ stands for the Gamma function. Surveying the history of publications on this topic reveals the limitations of conventional QFT in dealing with non-perturbative aspects of particle physics [3-4].

Let

$$S^{-1}(p) = \frac{(p^2 - m^2 + i0^+)^{(1+\gamma)}}{\gamma^{\mu}p_{\mu} + m} f(\frac{\Lambda}{m}) \approx (\gamma^{\mu}p_{\mu} - m + i0^+) f(\frac{\Lambda}{m}) \tag{9.4a}$$

$$f(\frac{\Lambda}{m}) = (\frac{i\Lambda}{m})^{\gamma} \tag{9.4b}$$

represent the inverse propagator entering (9.3). Relation (9.4) explicitly factors out the contribution of the standard inverse propagator $(\gamma^{\mu}p_{\mu} - m + i0^+)$ and the interpolating function $f(\Lambda/m) = (i\Lambda/m)^{\gamma}$ expressed in terms of two widely separated mass scales $m <<$ Λ and the fractional exponent γ.

This analysis is, however, not limited to the QED of charged fermions. Similar reasoning indicates that both scalar and gauge bosons of the Standard Model (SM) cannot be realistically approximated as excitations of free fields. In particular [1-2],

a) Higgs and Yang-Mills theories are *nonlinear dynamic models* which exhibit self-interaction, with the possible exception of the deep UV sector where they become ultra-weakly coupled or "trivial".

b) In general, the contribution of fermionic loops (and hypothetical new degrees of freedom arising beyond SM) cannot be fully balanced without invoking precise cancellation of competing diagrams ("*fine tuning*").

c) Although the SM is perturbatively renormalizable and free from anomalies, anomalous propagators and their corresponding behavior can still occur whenever conditions fall outside perturbation theory.

It is reasonable, on these grounds, to posit that inverse propagators acting at the boundaries of QFT are well approximated by their conventional form times a generic interpolating function, as in [5-6]

$$S_s^{-1}(p) \approx (p^2 - m^2 + i0^+) f(\frac{p^2}{p_0^2}) \text{ (scalars)} \quad (9.5a)$$

$$S_b^{-1}(p) \approx g_{\mu\nu}^{-1}(p^2 - m^2 + i0^+) f(\frac{p^2}{p_0^2}) \text{ (vector bosons, Feynman gauge)} \quad (9.5b)$$

$$S_f^{-1}(p) \approx (\gamma^\mu p_\mu - m + i0^+) f(\frac{p}{p_0}) \text{ (fermions)} \tag{9.5c}$$

Here, p_0 represents an arbitrary reference IR or UV momentum scale. In particular, the IR regime of massive scalar field theory is characterized by [5-6]

$$p_0 = p_{IR} < p < \Lambda \tag{9.6}$$

subject to the constraint

$$\frac{p_{IR}}{p} = \frac{p}{\Lambda} \Rightarrow p_{IR} = \frac{p^2}{\Lambda} \tag{9.7}$$

Near and below the lower limit of range (9.6), the scaling ratio (9.7) behaves as

$$\lim_{p \to p_{IR}} (\frac{p}{p_{IR}})^2 = 1 \quad (p \neq 0) \tag{9.8}$$

$$\lim_{p \to 0} (\frac{p}{p_{IR}})^2 = 0 \quad (p < p_{IR}) \tag{9.9}$$

Our goal is to further understand the structure and dynamic implications of the inverse propagator (9.5) using *fractional field theory* [7-10, 13-14]. The section is organized as follows: paragraph 9.2 introduces the concept of fractal propagator starting from the fractional Klein-Gordon equation; the connection between fractal propagators and FFT is presented in 9.3. Building on these premises, 9.4 derives the link between fractal propagators and classical gravity, whereby the latter emerges as long-range and ultra-weak excitation of the Higgs condensate.

9.2 THE FRACTAL PROPAGATOR CONCEPT

Consider the stationary fractional Klein-Gordon equation in one space dimension [11]

$$(D_x^{\beta} + m^2)\varphi = \rho(x) \tag{9.10}$$

where D_x^{β} is the differential operator of non-integer index β, $\rho(x)$ is a time-independent point source of strength g

$$\rho(x) = g\,\delta(x) \tag{9.11}$$

The choice $\beta = 2$ recovers the standard Klein-Gordon equation. The Green function can be evaluated taking the Laplace transform of (9.10), which leads to

$$G(m^2, p, \beta) = (p^{\beta} + m^2)^{-1} \tag{9.12}$$

If $\beta = 2 + \varepsilon$ with ε << 1, we obtain

$$G(m^2, p, 2+\varepsilon) = (p^{2+\varepsilon} + m^2)^{-1} \tag{9.13}$$

The solution of (9.10) may be explicitly expanded in Mittag-Leffler (ML) functions [11]

$$\varphi(x) = \sum_{k=0}^{[2+\varepsilon]} \{a_k x^{2+\varepsilon-k} E_{2+\varepsilon,3+\varepsilon-k}(-m^2 x^{2+\varepsilon}) + \int_0^x E_{2+\varepsilon,3+\varepsilon-k}(-m^2(x-x')^{2+\varepsilon}(x-x')^{1+\varepsilon}\rho(x')\,dx'\} \tag{9.14}$$

(9.14) represents a generalization of the Yukawa short-range solution in exactly four-dimensional spacetime ($\varepsilon = 0$)

$$\varphi_Y(x) = \frac{g}{4\pi}\frac{\exp(-mx)}{x} \tag{9.15}$$

where the presence of ML functions signals the onset of long-range spatial correlations in the behavior of the scalar field $\varphi(x)$ [11-12].

9.3 FRACTAL PROPAGATORS IN FRACTIONAL FIELD THEORY

Let us now take a detour and return to the conventional formulation of particle propagators in QFT [1-2]. The propagator for free massive spinless fields expressed in dimensionless form reads

$$S_s^*(\frac{p}{p_0}) = \frac{S_b(xp_0)}{p_0^2} = \int \frac{d^4p}{(2\pi)^4} \frac{p_0^2}{p_0^4} \exp(-i\frac{p}{p_0}xp_0) \frac{1}{p^2 - m^2 + i0^+} \tag{9.16}$$

or

$$S_s^*(\frac{p}{p_0}) = \int \frac{d^4(\frac{p}{p_0})}{(2\pi)^4} \exp(-ipx) \frac{1}{(\frac{p}{p_0})^2 - (\frac{m}{p_0})^2 + i0^+} \tag{9.17}$$

We introduce the inverse propagator in momentum space viz.

$$S_s^{-1}(\frac{p}{p_0}) = (\frac{p}{p_0})^2 - (\frac{m}{p_0})^2 + i0^+ \tag{9.18}$$

Using the line of arguments discussed in 9.2, the inverse propagator acting on the MFM is given by

$$S_s^{-1}(\frac{p}{p_0}, \varepsilon) = (\frac{p}{p_0})^{2(1+\varepsilon)} - (\frac{m}{p_0})^2 + i0^+ \tag{9.19}$$

(9.19) may be alternatively presented as

$$S_s^{-1}(\frac{p}{p_0},\varepsilon)=[(\frac{p}{p_0})^2-(m\frac{p_0^{\varepsilon-1}}{p^{\varepsilon}})^2+i0^+](\frac{p}{p_0})^{2\varepsilon} \tag{9.20}$$

We proceed with the assumption that the far IR scale is set by the cosmological constant, that is,

$$p_{IR}=\Lambda_{cc}^{1/4} \tag{9.21a}$$

Following [8, 13-14] as well as (5.4), dimensional regularization applied in the context of FFT requires the far IR scale ($\Lambda_{cc}^{1/4}$), the electroweak scale (M_{EW}) and the far UV scale fixed by the Planck mass $(\Lambda_{UV}=M_{Pl})$ to satisfy the constraint

$$\frac{\Lambda_{cc}^{1/4}}{M_{EW}}=\frac{M_{EW}}{\Lambda_{UV}}=O(\varepsilon) \tag{9.21b}$$

We are now set to explore the IR region of field theory ranging from the electroweak scale $p_0=M_{EW}<<\Lambda_{UV}$ to the far scale of cosmic distances $M_{EW}>p>>\Lambda_{cc}^{1/4}$. It makes sense to revisit the arguments previously made, apply the formalism to the Higgs sector of the Standard Model ($m=m_H$) and cast (9.20) as

$$S_H^{-1}(\frac{p}{M_{EW}},\varepsilon)=[(\frac{p}{M_{EW}})^2-(m_H\frac{M_{EW}^{\varepsilon-1}}{p^{\varepsilon}})^2+i0^+](\frac{p}{M_{EW}})^{2\varepsilon} \tag{9.22a}$$

Relation (9.22a) is well approximated by

$$S_H^{-1}(P,\varepsilon)\approx[P^2-M_H^2(\varepsilon)+i0^+]P^{2\varepsilon} \tag{9.22b}$$

where the "effective" momentum and "effective" Higgs mass are respectively defined as

$$P = \frac{p}{M_{EW}} \tag{9.23}$$

$$\frac{m_H}{p^{\varepsilon} M_{EW}} = M_H(\varepsilon) \tag{9.24}$$

A glance at (9.21a-b), (9.22a-b), and (9.5) reveals that the interpolating function

$$f(\frac{p}{M_{EW}}) = (\frac{p}{M_{EW}})^{2\varepsilon} \tag{9.25}$$

exhibits the following limiting behavior as $\varepsilon << 1$, $\varepsilon \neq 0$

$$p = O(M_{EW}) = O(m_H) \Rightarrow \lim_{\frac{M_{EW}}{\Lambda_{UV}} \to \varepsilon} (\frac{p}{M_{EW}})^{2\varepsilon} = 1 \tag{9.26}$$

$$p \leq O(\Lambda_{cc}^{1/4}) << M_{EW} \Rightarrow \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \to \varepsilon} (\frac{p}{M_{EW}})^{2\varepsilon} = 0 \ , \quad \text{if} \ p/M_{EW} << \varepsilon \tag{9.27}$$

It is instructive to note here that, consistent with the principles of effective field theory, in the far IR limit (9.27), the effective Higgs mass ($M_H(\varepsilon)$) of (9.22) diverges and naturally decouples from physics occurring at very large distances.

Combined use of (9.25) and (9.27) yields

$$\lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \to \varepsilon} f'(0) = \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \to \varepsilon} 2\varepsilon(\frac{p}{M_{EW}})^{2\varepsilon-1} \approx \lim_{\frac{\Lambda_{cc}^{1/4}}{M_{EW}} \to \varepsilon} \frac{2\varepsilon}{(\frac{p}{M_{EW}})} \approx \frac{2\varepsilon}{O(\varepsilon)} \approx O(1) \tag{9.28}$$

provided that p/M_{EW} does not fall too far below ε. We shall make use of (9.28) in the next paragraph.

9.4 CLASSICAL GRAVITY AS LONG-RANGE EXCITATION OF THE HIGGS CONDENSATE

An interesting proposal of [5-6] is that classical gravity may be modeled as long-range and ultra-weak excitation of the Higgs condensate. The approach developed here points in the same direction: The MFM favors the onset of long-range coupling and the emergence of interpolating functions of the type (9.4b) and (9.25) in the expression of propagators.

Following [5-6], the connection between Newton's constant (G_N) and Fermi's constant (G_F) is given by

$$G_N = \frac{p_{IR}^2}{4\pi f'(0) m_H^2} G_F \tag{9.29}$$

Substituting (9.21a-b) and (9.28) in (9.29) leads to

$$\boxed{G_N \sim 10^{-33} G_F} \tag{9.30}$$

in good agreement with currently known observational values of the two constants.

APPENDIX "A": THE HIERARCHY PROBLEM [1]

The EW symmetry of the SM is broken by a scalar field having the following doublet structure:

$$\Phi = \begin{pmatrix} G^{+} \\ \frac{1}{\sqrt{2}}[(H+\mathrm{V})+iG^{0}] \end{pmatrix} \tag{a.1}$$

Here, G^{+} and G^{0} represent the charged and neutral Goldstone bosons arisen from spontaneous symmetry breaking, H is the SM Higgs boson, $V \approx 246$ GeV is the Higgs vacuum expectation value. Symmetry breaking is caused by the Higgs potential, whose form satisfies the requirements of renormalizability and gauge-invariance:

$$V(\Phi,\Phi^{+}) = \mu_H^2\Phi^{+}\Phi + \lambda_0(\Phi^{+}\Phi)^2 \tag{a.2}$$

with $\lambda_0 \approx O(1)$ and $\mu_H^2 \approx O(M_{EW}^2)$. A vanishing quartic coupling ($\lambda_0 = 0$) represents the critical value that separates the ordinary EW phase from an unphysical phase where the Higgs field assumes unbounded values. Likewise, the coefficient μ_H^2 plays the role of an order parameter whose sign describes the transition between a symmetric phase and a broken phase. Minimizing the Higgs potential yields an expectation value given by:

$$\mathrm{V}^2 = -(\mu_H^2/\lambda_0) \tag{a.3}$$

where the physical mass of the Higgs is:

$$m_H^2 = 2\lambda_0\mathrm{V}^2 = -2\mu_H^2 \tag{a.4}$$

The renormalized mass squared of the Higgs scalar contains two contributions:

$$\mu_H^2 = \mu_{0,H}^2 + \Delta\mu^2 \tag{a.5}$$

in which $\mu_{0,H}^2$ represents the ultraviolet (bare) value. This mass parameter picks up quantum corrections $\Delta\mu^2$ that depend quadratically on the ultraviolet cutoff Λ_{UV} of the theory. Consider for example the contribution of radiative corrections to μ_H^2 from top quarks. The complete one-loop calculation of this contribution reads:

$$\Delta\mu^2 = \frac{N_c\lambda_t^2}{16\pi^2}[-2\Lambda_{UV}^2 + 6M_t^2\ln(\frac{\Lambda_{UV}}{M_t}) + ...] \quad (a.6)$$

in which λ_t and M_t are the Yukawa coupling and mass of the top quark. If the bare Higgs mass is set near the cutoff $\mu_{0,H}^2 = O(\Lambda^2) = O(M_{Pl}^2)$, then $\Delta\mu^2 \approx -10^{35}$ GeV². This large correction must precisely cancel against $\mu_{0,H}^2$ to protect the EW scale. This is the root cause of the hierarchy problem, which boils down to the implausible requirement that $\mu_{0,H}^2$ and $\Delta\mu^2$ should offset each other to about 32 decimal places.

APPENDIX "B": LIMITATIONS OF PERTURBATIVE RENORMALIZATION AND THE CHALLENGES OF THE SM [1, 2]

In contrast with the paradigm of effective Quantum Field Theory (EFT), realistic RG flows approaching fixed points are neither perturbative nor linear. The object of this Appendix section is to show that that overlooking these limitations is necessarily linked to many unsolved puzzles challenging the SM. In particular, as discussed in chapters 3 to 5, the analysis of nonlinear attributes of the RG flow equations near the EW scale can account for the full flavor and mass spectrum of the SM and brings closure to the "naturalness" puzzle without impacting the cluster decomposition principle of EFT.

B1. INTRODUCTION

In his 1979 seminal paper on "Phenomenological Lagrangians", Steven Weinberg has formulated the fundamental principles that any sensible EFT must comply with in order to successfully explain the physics of the subatomic realm: Quantum Field Theory (QFT) has no content besides *unitarity*, *analyticity*, *cluster decomposition* and *symmetries*. This conjecture implies that, in order to compute the *S*-matrix for any field theory below some scale, one must use the most general effective Lagrangian consistent with these principles expressed in terms of the appropriate asymptotic states.

Closely related to Weinberg's conjecture are two key aspects of EFT that deal with the separation of heavy degrees of freedom from the light ones. One is the *Decoupling Theorem* (Appelquist-Carrazone) stating that the effects of heavy particles go into local terms in a field theory, either renormalizable couplings or in non-renomalizable effective interactions suppressed by powers of the heavy scale. The other is *Wilson's Perturbative Renormalization Program* who teaches how to separate the degrees of freedom above and below a given scale and then to integrate out all the high-energy effects and form a low-energy field theory with the remaining degrees of freedom below the separation scale.

The idea of scale separation in EFT is typically illustrated by considering the perturbative expansion of amplitudes in powers of momenta Q over a large scale Λ_{UV}, the latter setting the upper limit of validity for the EFT

$$M(Q/\mu, g_n, \Lambda_{UV}) = \sum_{\rho} (Q/\Lambda_{UV})^{\rho} f(Q/\mu, g_n) \qquad \text{(b.1)}$$

Here, μ represents the RG scale, g_n are the low-energy couplings, the function f is of order unity $O(1)$ (expressing the "*naturalness*" of the theory) and the summation index ρ is bounded from below. The contribution of the large scale is naturally suppressed as $\Lambda_{UV} >> Q$.

In this work we re-examine Wilson's Renormalization ideas as traditionally viewed from the standpoint of EFT. The motivation stems from the fact that, as pointed out in the Introduction section, although a fully consistent and well supported theoretical framework, the SM continues to be plagued by numerous conceptual challenges. Our basic premise is that realistic Renormalization Group (RG) flows approaching fixed points cannot be restricted to be either *perturbative* or *linear*. Here we point out that imposing these upfront restrictions is inevitably linked to the many challenges left unanswered within the SM. In particular, the second paragraph surveys the general construction and limitations of the RG program, with emphasis on the conclusion that non-renormalizable interactions vanish at the low energy scale.

B2. LIMITATIONS OF THE RG PROGRAM

As local QFT residing on Minkowski spacetime is expected to break down at very short distances due to (at the very least) quantum gravity effects, any physically sensible theory must include a high-energy cutoff (Λ_0). The *continuum limit* is defined by a cutoff approaching infinity ($\Lambda_0 \to \infty$). To simplify the presentation, consider a local scalar field theory in four dimensional spacetime where all field modes above some arbitrary momentum scale $\Lambda < \Lambda_0$ have been integrated out. The Lagrangian of such an effective theory assumes the form

$$L_\Lambda = \sum_n a_n(\Lambda) O_n(\varphi_\Lambda) \tag{b.2}$$

where $O_n(\Lambda)$ represent the set of local field operators, including their spacetime derivatives, and $a_n(\Lambda)$ the set of coupling parameters. If $O_n(\Lambda)$ have mass dimensions $4-d_n$, $a_n(\Lambda)$ carry mass dimensions d_n and one can cast all couplings in a dimensionless form as in

$$g_n(\Lambda) = a_n(\Lambda)\Lambda^{-d_n} \tag{b.3}$$

The behavior of local operators $O_n(\Lambda)$ depends on their mass dimensions: relevant operators correspond to $d_n > 0$, marginal operators to $d_n = 0$ and irrelevant operators to $d_n < 0$. All mass dimensions are assumed to be scale independent. Since Λ is arbitrary, we may fix the dimensionless couplings (b.3) at some reference scale chosen to lie in the deep ultraviolet region and yet far enough to the cutoff, say $\Lambda_{UV} < \Lambda_0$

$$\overline{g_n} = g_n(\Lambda_{UV}) \tag{4}$$

The flow of the coupling parameters with respect to a sliding RG scale $\mu < \Lambda_{UV}$ is then described by the system of partial differential equations

$$\mu \frac{\partial}{\partial \mu} g_n(\mu) = \beta_n(g_n; \mu/\Lambda_{UV}) \tag{b.5}$$

The above flow equations imply that the couplings measured at the sliding scale μ depend on the high-energy parameters $\overline{g_n}$ and on the ratio μ/Λ_{UV} as in

$$g_n(\mu) = g_n(\overline{g_n}; \mu/\Lambda_{UV}) \tag{b.6}$$

We assume below that there are N relevant and marginal operators with mass dimensions less than or equal to 4. The operators belonging to this set are denoted by the Roman indices $a, b, \dots$, whereas the irrelevant operators with dimension greater than 4 are indicated by Greek indices $\alpha, \beta \dots$. The Roman characters $m, n, r \dots$ describe the general set of operators and couplings.

It can be shown that in the regime of *weakly coupled perturbation theory*, the RG flow (b.5) projects an arbitrary initial surface in the UV coupling space $\{\overline{g_n}\}$ to a N-dimensional surface of $\{g_n(\mu)\}$, a given point of which is uniquely specified by N low-energy parameters, up to corrections that decay as inverse powers of the ratio μ/Λ_{UV}. The proof relies exclusively on a *linear stability analysis* of flow equations (b.5) and leads to the following relationships, valid for $\mu << \Lambda_{UV}$

$$\boxed{\delta g_\alpha(\mu) \sim G_{\alpha a} G_{ab}^{-1} \delta g_b(\mu) + O(\delta^* g_\alpha)} \tag{b.7}$$

where

$$\delta^* g_\alpha \sim \left(\frac{\mu}{\Lambda_{UV}}\right)^{|d_\alpha|} \tag{b.8}$$

As mentioned above, α denotes the index of irrelevant couplings and operators present in the theory. Here, $\delta^* g_\alpha$ represents the set of first order variations in the irrelevant couplings

$$\delta^* g_\alpha(\mu) = \delta g_\alpha(\mu) - G_{\alpha a} G_{ab}^{-1} \delta g_b(\mu) \tag{b.9}$$

The matrix $G_{nm}(\mu)$ defines the variation of the low-energy parameters g_n under variations of the initial high-energy parameters $\overline{g_m}$ specified by (b.4), that is,

$$G_{nm}(\mu) = \frac{\partial g_n(\mu)}{\partial \overline{g_m}(\mu)} \tag{b.10}$$

The finite $N \times N$ sub-matrix G_{ab} contains rows and columns restricted to the marginal and relevant couplings. *Relation (b.7) states that the contribution of irrelevant couplings and operators at low energy (indexed by* α *) may be entirely absorbed in variations of the marginal and relevant couplings (indexed by b).*

Despite being rigorously derived, (b.7) is founded on a set of *simplifying assumptions* which disqualifies it from being a universal result. In particular,

1) The matrix G_{ab} is constrained to be *nonsingular*, which fails to be true for isolated sets of measure zero in coupling space.

2) The theory is considered *weakly coupled* to make the perturbation analysis applicable.

3) The linear stability of the flow equations is assumed to hold true in general. With reference to planar flows, this is a legitimate approximation only if the fixed points do not fall in the category of *borderline equilibria* (such as centers, degenerate nodes, stars or non-isolated attractors or repellers).

4) The flow equations are assumed to correspond to Markov processes, that is, they are *immune to memory effects*.

5) Bound states are excluded from this approach, as they require an entirely *non-perturbative treatment*.

It is somehow surprising that many QFT textbooks do not explicitly point out the limitations that these assumptions place on the validity of field theories in general. The widespread belief is that they do not appear to directly impact the cluster decomposition principle and all SM predictions up to the low-TeV scale probed by the LHC. However, in light of all unsettled questions confronting the SM, one cannot help but wonder if some important piece of the puzzle is not lost in overlooking these limitations. For example, over past decades the prevailing consequence of the concept of "*naturalness*" for model building has been the cancellation of quadratic divergences to the SM Higgs mass (see Appendix A). According to this paradigm, the SM itself is an unnatural theory, mandating new physics somewhere near the low-TeV scale. At the same time the LHC, flavor physics, electroweak precision results and evaluation of the electron dipole moment all point to the absence of any new phenomena in this range, which is however necessary to accommodate the observation of both neutrino oscillations and cold Dark Matter.

It seems that a paradigm shift is clearly needed to understand both the SM and the physics lying beyond it. Tackling this challenge from a novel perspective rooted on the RG program is indeed the main motivation behind our book.

We end this Appendix section with the key observation that, since the continuum field theory is only an "effective" space-time model, the effects induced by the dimensional parameter $\varepsilon = 4 - D$, with $\varepsilon << 1$, *are not perceivable* in the computation of scattering amplitudes (b.1) at the SM scale. With reference to (b.1), the condition $\varepsilon << 1$ is equivalent to setting $\mu = \mu_{SM} = O(Q) << \Lambda_{UV}$ and the contribution of ε becomes strongly suppressed by the power expansion (b.1). *As a result, the cluster decomposition principle of EFT remains insensitive to the emergence of fractal space-time near or above the SM scale (* $\mu \geq \mu_{SM} = O(M_{EW})$ *).*

APPENDIX "C": A PRIMER ON FRACTALS AND MULTIFRACTALS [1]

We highlight here few basic concepts and terminology pertaining to fractals and multi-fractals. Fractals are geometrical objects with non-integer dimensions that display self-similarity on all scales of observation. The concept of *dimension* plays a key role in the geometry of fractal sets. It is customary to characterize fractals by an ensemble of three dimensions, namely:

1) The Euclidean dimension "$D = 1, 2, 3...$" represents the dimension of the space where the object resides and is always an integer.

2) The topological dimension "$d_T \le D$" describes the dimensionality of continuous primitive objects such as points, curves, surfaces or volumes ($d_T = 0,1,2,3$ in ordinary four-dimensional spacetime).

3) The definition of the fractal (or *Hausdorff*) dimension is as follows: Cover the fractal object by $d-$dimensional balls of radius "Δ" and let "$N(\Delta)$" be the minimum number of balls needed for this operation. The fractal dimension "D_H" satisfies the inequality $d_T \le D_H \le D$ and is given by

$$\lim_{\Delta \to 0} N(\Delta) = \Delta^{-D_H} \quad \text{(c.1)}$$

leading to

$$D_H = \lim_{\Delta \to 0} [\frac{\log N(\Delta)}{\log \Delta^{-1}}] \quad \text{(c.2)}$$

Many of the self-similar structures in fractal geometry are built recursively, a typical example being the *Cantor set*. To construct a Cantor set in one dimension ($D = 1$), take a line segment called the *generator*, split it into thirds and remove the middle third. Iterate this process arbitrarily many times. One is left with a countable set of isolated points having a non-integer fractal dimension D_H, with $d_T = 0 \le D_H \le D = 1$. A simple Cantor set generated from segments of equal length is defined by a single scaling factor $r = 1/3 < 1$. By contrast, more general fractals (such as *multifractals*) can be created using generator segments of different scaling factors $r_i < 1$, $i = 1,2,...,N$ satisfying the closure relation

$$\sum_{i=1}^{N} r_i^{D_H} = 1 \tag{c.3}$$

Many strange attractors of nonlinear dynamical systems represent multifractals and are typically characterized by a continuous spectrum of Hausdorff dimensions.

APPENDIX "D": ON NON-INTEGRABILITY AND THE ASYMPTOTIC BREAKDOWN OF PERTURBATIVE FIELD THEORY

There are several instances where non-analytic functions and non-integrable operators are deliberately excluded from perturbative QFT and RG to maintain internal consistency of both frameworks. Here we briefly review these instances and suggest that they may be a portal to an improved understanding of the asymptotic sectors of QFT and the SM.

D1. THE FEYNMAN-DYSON INTEGRALS

It is well known that the standard formulation of perturbative QFT relies on the Feynman – Dyson series of integrals [1, 2]

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 ... \int_{-\infty}^{\infty} dt_n \times T\{H_I(t_1) H_I(t_2) ... H_I(t_n)\} \tag{d.1}$$

where the integrand consists of the time-ordered product of the interaction Hamiltonian $H_I(t)$. The interaction Hamiltonian is typically written as

$$H_I(t) = \int d^3x \, H(\mathbf{x}, t) \tag{d.2}$$

in which $H(\mathbf{x},t)$ is a polynomial whose terms are local functions of the annihilation and creation fields viz.

$$\psi_l^{(+)}(x) = \int d^3p \sum_{\sigma,n} \exp(ipx)\, u_l(\mathbf{p},\sigma,n)\, a(\mathbf{p},\sigma,n) \qquad \text{(d.3)}$$

$$\psi_l^{(-)}(x) = \int d^3p \sum_{\sigma,n} \exp(-ipx)\, v_l(\mathbf{p},\sigma,n)\; a^{+}(\mathbf{p},\sigma,\bar{n}) \qquad \text{(d.4)}$$

$$\psi_l(x) = \sum_{\sigma,n} [\psi_l^{(+)}(x) + \psi_l^{(-)}(x)] \qquad \text{(d.5)}$$

Here, $\psi_l^{(+)}(x)$ and $\psi_l^{(-)}(x)$ annihilate particles and create antiparticles, respectively, $\mathbf{p}$ represents the three-momentum, σ the z-projection of the spin, n and $\bar{n}$ label the number of particle and antiparticle species, respectively. Lorentz transformation properties of the fields and one-particle states, along with the constraint that fields commute at space-like separations, fix entirely the form of the coefficients u_l and v_l.

A cornerstone requirement of QFT is the *cluster decomposition principle* (CDP) which states that distant experiments yield uncorrelated outcomes. In particular, CDP protects low-energy physics from short-distance perturbations. CDP requires that the interaction Hamiltonian be formulated as a power series in the creation and annihilation operators, which are *sufficiently smooth* functions of the momenta. This condition is automatically satisfied by an interaction Hamiltonian having the form (d.2) [3]. While there is widespread consensus on the compelling success of perturbative QFT in particle physics and condensed matter, restricting the analysis to sufficiently smooth Hamiltonians is likely to produce unrealistic approximations in future cases of interest. Recent years have consistently shown that many nonlinear dynamical systems display *non-smooth interactions, bifurcations, limit cycles, strange attractors, non-Gaussian noise or multifractal properties*. To give only one example, consider the class of *non-integrable systems* arising in the context of the three-body problem, chaotic oscillators, KAM theory, Henon-Heiles potential, kicked rotor, turbulent flows and so on. One is motivated to ask: *What happens if the interaction Hamiltonian is allowed to contain non-smooth contributions in the structure of creation and annihilation operators?* We discuss this topic next.

D2. NON-PERTURBATIVE EFFECTS OF THE RG FLOW

One plausible scenario is that the non-smooth contributions emerge at the low-energy scale of effective QFT as residual non-perturbative effects of the RG flow [4, 5]. To fix ideas, we follow [2] and refer to the framework of *effective field theories* (EFT). In general, the construction of EFT is based on the so-called "momentum-shell" approach, which consists of a two-step procedure:

a) change of functional variables of integration in the path integral formulation of the theory,

b) perform partial evaluation of the modified path integral whereby short-wavelength fields are integrated out in the absence of external currents.

The core assumption of both CDP and EFT holds that the Lagrangian built from the remaining "coarse-grained" fields supplies exact results for the *n*-point amplitudes. Let $\varphi_n,\ n=1,2,...N$ denote the complete set of short-wavelength fields characterizing the dynamics of the theory at some running high-energy scale $\Lambda < \Lambda_{UV}$, in which Λ_{UV} stands for the ultraviolet cutoff. The new set $m = 1, 2, 3 \ldots M$ of "coarse-grained" fields are defined through

$$\Phi_m = f_m(\varphi_n; \Lambda) \tag{d.6}$$

where $M < N$ and the "coarse-graining" functions $f_m(\ldots)$ are typically non-invertible. If $L(\varphi_n)$ represents the Euclidean Lagrangian associated with the short-wavelength fields, the effective Lagrangian corresponding to the "coarse-grained" fields (d.6) takes the form

$$\exp[-\int d^4 L_{eff}(\Phi_m)] = \int \prod_n D\varphi_n \, \delta[\Phi_m - f_m(\varphi_n; \Lambda)] \exp[-\int d^4 L(\varphi_n)] \tag{d.7}$$

The meaning of (d.7) is that the original "microscopic" Lagrangian can be safely factored out when computing the *n*-point amplitudes of Φ_m in the presence of external currents J_m. This is because the generating functional for Φ_m can be expressed in a form that does not preserve any memory of the microscopic fields, that is,

$$Z(J_m) = \int \prod_m D\Phi_m \exp[-\int d^4 x L_{eff}(\Phi_m) + \int d^4 x \, J_m(x) \Phi_m(x)] \tag{d.8}$$

The functions $f_m(\ldots)$ are required to be smooth in order for the effective Lagrangian (d.7) to be expanded in multi-monomials of local products of Φ_m. It is also readily seen from (d.7) that the effective Lagrangian becomes ill-defined if $L(\varphi_n)$ is either non-smooth or non-integrable. One cannot arbitrarily discard this possibility in the near or far Terascale sector of high-energy physics *or* in a dynamic environment that no longer comply with the conditions of equilibrium statistical physics [6]. Likely plausible is the case where "coarse graining" is partially successful, only part of the EFT survives and some residual non-smooth contributions persist at the EFT scale.

D3. THE DAMPING FUNCTION IN THE "MOMENTUM-SHELL" INTEGRATION SCHEME

These considerations can also directly impact the basis of the "momentum-shell" approach. The "momentum-shell" approach turns out to be invalid from an analytical point of view as sharp momentum scale yields *singular terms* in taking derivatives in the RG flow equations [2]. To correct this deficiency, it is necessary to introduce a suitable *damping function* $D(k^2/\Lambda^2)$ whose role is to "blur" the sharp momentum scale and to suppress the loop integrals arisen from internal propagators that exceed this scale ($|k| > \Lambda$). The damping function "coarse-grains" the free part of the effective Lagrangian in momentum space viz.

$$S_0[\Phi,\Lambda] = \int \frac{1}{2}\Phi(k)(k^2+m^2)D(\frac{k^2}{\Lambda^2})\Phi(-k)\frac{d^4k}{(2\pi)^4} \tag{d.9}$$

The damping function is required to be strictly *analytic* in order to maintain the locality property of the theory: in particular, it has to ensure that the effective Lagrangian at any scale can be expanded into an infinite sum of local terms, where each term includes products of fields and their derivatives defined at *single* space-time locations.

APPENDIX "E": CONSERVATION LAWS FROM THE MINIMAL FRACTAL MANIFOLD

The aim of this Appendix section is to uncover a tentative link between the MFM and the symmetry principles underlying QFT.

The momentum norm of a free relativistic particle is given by

$$P^2 = p^\mu p_\mu = (p^0)^2 - \mathbf{p}^2 = (p^0 - |\mathbf{p}|)(p^0 + |\mathbf{p}|) = m^2 \quad \text{(e.1)}$$

in which the rest-frame mass can be factored out as in

$$m^2 = m_- m_+ = (p^0 - |\mathbf{p}|)(p^0 + |\mathbf{p}|) \quad \text{(e.2)}$$

Referring to (4.21)-(4.23) and using the above relation yields

$$r_i^2 = \frac{\varepsilon_i}{\varepsilon_0} = \frac{m_i^2}{M_{EW}^2} = \frac{(m_- m_+)_i}{M_{EW}^2} \quad \text{(e.3)}$$

(e.3) hints to a similar factorization of the dimensional parameter ε_i introduced in (4.21)-(4.23), namely

$$\varepsilon_i = [\varepsilon_{i(-)}\varepsilon_{i(+)}]^{1/2} = \{[(\varepsilon_i^0)^2 - \boldsymbol{\varepsilon}_i^2][(\varepsilon_i^0)^2 + \boldsymbol{\varepsilon}_i^2]\}^{1/2} \quad \text{(e.4)}$$

Here, ε_i^0 is the replica of the temporal component whereas $\boldsymbol{\varepsilon}_i^2$ replicates the norm of the spatial component of the four momentum vector. Therefore,

$$\frac{(m_-)_i^2}{M_{EW}^2} \Rightarrow (\varepsilon_i^0)^2 - \boldsymbol{\varepsilon}_i^2 \;,\; \frac{(m_+)_i^2}{M_{EW}^2} \Rightarrow (\varepsilon_i^0)^2 + \boldsymbol{\varepsilon}_i^2 \quad \text{(e.5)}$$

It is well known that Lorentz symmetry applied to a free relativistic particle enables one to arbitrarily select various inertial frames of reference (and different corresponding components of the four-momentum) by holding the rest-mass m invariant. Likewise, in light of the above relations (e.3) to (e.5), one is free to arbitrarily choose ε_i^0 and $\boldsymbol{\varepsilon}_i^2$, provided that ε_i stays unchanged.

These observations suggest that the four-momentum conservation can be mapped to the requirement of holding the fractal dimension ε_i constant. In particular, they hint that there is an intriguing correspondence between the *Lorentz transformation of inertial frames and the transformation properties of dimensional components* ε_i^0 *and* $\boldsymbol{\varepsilon}_i^2$. Same reasoning goes for the electric charge, as a result of (3.14*). Conservation of the electric charge by "rotations" in U(1) space is equivalent to the requirement of holding the dimensional parameter* ε *constant.*

In summary, we find that there is an unforeseen *duality* between spacetime and gauge symmetries, on the one hand, and the invariance of dimensional parameter $\varepsilon << 1$ on the other.

10. CONCLUSIONS

In a letter to a friend written one year prior to his death, Einstein remarked:

"*I consider it quite possible that physics cannot be based on the field concept, that is, on continuous structures*".

In hindsight, given the difficulties of quantizing classical gravity on discrete space-time models along with the persistent lack of compelling evidence for Quantum Gravity [1], one cannot help but wonder if Einstein had a visionary insight on *fractal geometry*, more than twenty years before Mandelbrot's seminal work of 1975. Unlike the familiar continuum or discontinuous objects of our everyday experience, fractals are ubiquitous geometrical structures characterized by *non-integer dimensions*, *non-differentiability* in the conventional sense and *self-similarity* on all scales of observation. It is widely recognized nowadays that the mathematics of chaos, nonlinear dynamics and multifractals has found a broad range of applications in many branches of human endeavor. Seen in this context, *fractional field theory* has recently surfaced as a rapidly evolving field of research in theoretical physics. In a nut-shell, it amounts to a genuine attempt of carrying QFT, RG and the SM beyond perturbation theory and equilibrium statistical physics into the realm of complexity and dynamically evolving structures.

Since the high-energy theory is built from nonlinearly interacting operators, the underlying principle of this book is that the *universal behavior of nonlinear dynamical systems* must play a critical role in shaping the physics of the SM. An unavoidable corollary of this principle is that the basic premises of Wilson's program on the behavior of RG flows near fixed points lose their generality. As detailed throughout the main text, evaluating Wilson's RG theory from this vantage point necessarily leads to the concept of *minimal fractal manifold* (MFM) and sets the stage for a novel perspective on the gauge structure and dynamics of the SM. Our preliminary findings can be summarized as follows:

- The continuum limit of QFT is a weak manifestation of *fractal geometry*.

- Nonlinear behavior of RG flow equations is able to account for the *self-similar structure* of SM parameters, including its gauge and flavor content.
- In close proximity to the electroweak scale, the ordinary four-dimensional space-time turns into a MFM which makes the SM a *self-contained multi-fractal set*.
- The concept of MFM can account for the *dynamic generation of mass scales* in field theory.
- The Higgs scalar emerges as *condensate of gauge bosons* on the MFM.
- The "*hierarchy problems*" associated with the SM and the cosmological constant are solved in the context of the MFM by the natural separation of the electroweak scale, far infrared scale and the far ultraviolet scale.
- *Charge quantization* and the topological underpinning of *quantum spin* can be understood as direct outcomes of the MFM.
- The MFM concept enables a natural link to the asymptotic manifestation of *non-commutative field theory* and *q-deformed field theory*. In addition, it suggests a straightforward explanation on fermion chirality and the breaking of parity and temporal symmetry in electroweak interactions [2, 3].
- Classical gravity emerges as *dual manifestation* of field theory on the MFM and may be understood as *long-range and ultra-weak excitation of the Higgs condensate*.
- The MFM concept opens the door for the possibility of *exotic states of matter* and offers novel viewpoints on the dynamics and composition of the Dark Sector [4-6].

LIST OF ABBREVIATIONS

SM = Standard Model of elementary particle physics

LHC = Large Hadron Collider

QFT = Quantum Field Theory

EFT = effective Quantum Field Theory

RG = Renormalization Group

MFM = Minimal Fractal Manifold

EW = Electroweak

LGW = Landau-Ginzburg-Wilson

UV = ultraviolet

IR = infrared

FP = fixed point

QCD = Quantum Chromodynamics

QED = Quantum Electrodynamics

QG = Quantum Gravity

GUT = Grand Unified Theory

CFT = Conformal Field Theory

NC = Non-Commutative

NCT = nearly conformal transformation

CDP = cluster decomposition principle

REFERENCES

Introduction:

[1] http://home.web.cern.ch/about/updates/2013/03/new-results-indicate-new-particle-higgs-boson

[2] http://arxiv.org/abs/1503.07589 and http://pdg.lbl.gov/2013/reviews/rpp2013-rev-higgs-boson.pdf

[3] http://arxiv.org/abs/0801.2562

[4] http://arxiv.org/abs/1005.1676

[5] http://www.lupm.univ-montp2.fr/users/qcd/econf13/hepmad13_talks/sturdy-hepmad13.pdf

[6] Sornette D., "*Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-organization and Disorder*", (2006), Springer Series on Complex Systems.

[7] Goldfain E., "*Fractional Field Theory and Physics Beyond the Standard Model*", *Prespacetime Journal*, **3**(5), (2012), pp. 435-438.

[8] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", Quantum Matter, **3**(3), (2014), pp. 256-263.

[9] http://www.mehtapress.com/toc/article.php?id=258

[10] http://www-tkm.physik.uni-karlsruhe.de/~mirlin/multifrac.pdf

[11] http://mesoimage.grenoble.cnrs.fr/IMG/pdf/delande.pdf

[12] http://www.internonlinearscience.org/upload/papers/20110301080249197.pdf

[13] http://link.springer.com/chapter/10.1007/978-3-7643-8736-5_15

[14] For a relevant status update, see https://medium.com/starts-with-a-bang/the-road-less-traveled-to-quantum-gravity-594ac38e4544

[15] http://ptp.oxfordjournals.org/content/104/4/887.full.pdf+html

Section 1:

[1] http://hitoshi.berkeley.edu/230A/regularization.pdf

[2] Zinn-Justin J., "*Quantum Field Theory and Critical Phenomena*", (2002), Oxford University Press.

[3] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[4] Kaku, M., "*Quantum Field Theory, a Modern Introduction*", (1993), Oxford University Press.

Section 2:

[1] Maggiore M., "*A Modern Introduction to Quantum Field Theory*", (2006), Oxford University Press.

[2] D. Sornette, "*Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-organization and Disorder*", (2006), Springer Series on Complex Systems.

[3] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[4] Goldfain E., "*Fractional Field Theory and High-Energy Physics: New Developments*" in Horizons in World Physics, 279, (2013), Nova Science Publishers, 69-92.

[5] Goldfain E., "*Fractional Field Theory and Physics Beyond the Standard Model*", *Prespacetime Journal*, 3(5), (2012), pp. 435-438.

[6] Goldfain E., "*Reflections on the Future of Quantum Field Theory*", in Vision of Oneness, I. Licata and A. Sakaji, Editors, Aracne Editrice, (2011), pp. 273-312.

[7] Goldfain E., "*Non-equilibrium Theory, Fractional Dynamics and Physics of the Terascale Sector*" in *New Developments in the Standard Model*, (2012), Nova Science Publishers, pp. 41-74.

[8] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", Quantum Matter, **3**(3), (2014), pp. 256-263.

[9] Goldfain E., "*Feigenbaum Attractor and the Generation Structure of Particle Physics*", Intl. J. Bifurcation Chaos 18, (2008), pp. 891.

Section 3:

[1] Schwartz M. D., "*Quantum Field Theory and the Standard Model*", (2013), Cambridge University Press.

[2] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[3] Goldfain E., "*Chaotic Dynamics of the Renormalization Group Flow and Standard Model Parameters*", (2007), Intl. Journal of Nonlinear Science **3** (3), pp.170-180.

[4] http://arxiv.org/pdf/1011.3643v3.pdf

[5] http://arxiv.org/abs/hep-th/0310213

[6] http://arxiv.org/pdf/1304.2821v2.pdf

[7] http://arxiv.org/pdf/0809.1003v5.pdf

[8] http://www.sciencedirect.com/science/article/pii/S0167278997002868

[9] Strogatz H. Steven, "*Nonlinear Dynamics and Chaos*", (2000), Westview Press.

[10] Creswick R. J., Farach H.A, Poole C. P., "*Introduction to Renormalization Group Methods in Physics*", (1992), John Wiley & Sons.

[11] Peitgen H. O., Jürgens H., Saupe D., "*Chaos and Fractals, New Frontiers of Science*", (1992), Springer-Verlag.

[12] Donoghue J. F., Golowich E., Holstein B. R., "*Dynamics of the Standard Model*", (1994), Cambridge Univ. Press.

[13] http://www-library.desy.de/preparch/desy/thesis/desy-thesis-08-004.pdf

Section 4:

[1] http://arxiv.org/abs/1309.7296

[2] http://arxiv.org/abs/1305.3497

[3] http://arxiv.org/pdf/1210.2754v3.pdf

[4] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", (2014), Quantum Matter, 3(3), pp. 256-263.

[5] Goldfain E., "*Feigenbaum Attractor and the Generation Structure of Particle Physics*", (2008), Int. J. Bifurcation Chaos 18, pp. 891.

[6] Goldfain E., "*Reflections on the Future of Quantum Field Theory*", in Vision of Oneness, I. Licata and A. Sakaji, Editors, (2011), Aracne Editrice, pp. 273-312.

[7] Goldfain E., "*Non-equilibrium Theory, Fractional Dynamics and Physics of the Terascale Sector*" in New Developments in the Standard Model, (2012), Nova Science Publishers, pp. 41-74.

[8] Tarasov V., "*Electromagnetic Fields on Fractals*", (2006), Mod. Phys. Lett. A, 21, p. 1587.

[9] Ryder L., "*Quantum Field Theory*", (1996), Cambridge Univ. Press.

[10] Creswick R. J., Farach H.A, Poole C. P., "*Introduction to Renormalization Group Methods in Physics*", (1992), John Wiley & Sons.

[11] Goldfain E., "*Chaotic Dynamics of the Renormalization Group Flow and Standard Model Parameters*", (2007), Intl. Journal of Nonlinear Science 3 (3), pp.170-180.

[12] http://arxiv.org/pdf/1305.4208v1.pdf

[13] http://arxiv.org/pdf/1506.00962v1.pdf

[14] http://arxiv.org/pdf/1407.3683v2.pdf

[15] Goldfain E., "*Fractional dynamics and the TeV regime of field theory*", (2008), Comm. in Nonlinear Science and Numer. Simul. 13(3), pp. 666-676.

[16] Goldfain E., "*Fractional dynamics and the Standard Model for particle physics*", (2009), Comm. in Nonlinear Science and Numer. Simul. 13(7), pp. 1397-1404.

[17] Kaku, M., "*Quantum Field Theory, a Modern Introduction*", (1993), Oxford University Press.

Section 5:

[1] Schwartz M. D., "*Quantum Field Theory and the Standard Model*", (2013), Cambridge University Press.

[2] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[3] http://arxiv.org/pdf/hep-th/0003004v2.pdf

[4] http://arxiv.org/pdf/math-ph/0604028.pdf

[5] http://www.itp.uni-hannover.de/saalburg/Lectures/wiese.pdf

[6] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", Quantum Matter, **3**(3), (2014), pp. 256-263.

[7] Grosse H., Wulkenhaar R., "*Regularization and Renormalization of Quantum Field Theories on Non-Commutative Spaces*", Journal of Nonlinear Math. Phys., 11, (2004), 9-20 Bialowieza XXI, XXII.

[8] Goldfain E., "*Emergence of the Electroweak Scale from Fractal Spacetime*", Prespacetime Journal, 4(9), (2013), pp. 923-926.

[9] Goldfain E., "*Multifractal Theory and Physics of the Standard Model*", Prespacetime Journal, 5(7), (2014), pp. 554-565.

[10] Goldfain E., "*Fractal Propagators and the Asymptotic Sectors of Quantum Field Theory*", Prespacetime Journal, 5(8), (2014), pp. 712-719.

[11] Goldfain E., "*Fractional Field Theory and Physics Beyond the Standard Model*", Prespacetime Journal, 3(5), (2012), pp. 435-438.

[12] Goldfain E., "*Fractal Space-time as Tentative Solution for the Cosmological and Coincidence Problems*", Prespacetime Journal 4(8), (2013), pp. 742-747.

[13] Goldfain E., "*Fractional Field Theory and High-Energy Physics: New Developments*" in Horizons in World Physics, 279, Nova Science Publishers, (2013), pp. 69-92.

[14] http://arxiv.org/pdf/1007.1084v1.pdf

[15] Herrmann, R., "*Fractional Calculus: an Introduction for Physicists*", World Scientific, (2011), Singapore.

[16] Tao Y., "*The Validity of Dimensional Regularization Method on Fractal Spacetime*", (2013), Journal of Applied Mathematics, Hindawi.

[17] http://arxiv.org/pdf/1210.2754v3.pdf

[18] http://arxiv.org/pdf/1107.5041v4.pdf

[19] Goldfain E., "*Emergence of the Electroweak Scale from Fractal Spacetime*", Prespacetime Journal, **4**(9), (2013), pp. 923-926.

[20] http://www.ccsem.infn.it/issp2013/newtalent/poster_yerokhin.pdf

[21] http://arxiv.org/pdf/gr-qc/0602037v3.pdf

[22] http://arxiv.org/pdf/hep-th/0009139.pdf

[23] http://arxiv.org/pdf/gr-qc/9310026.pdf

[24] http://arxiv.org/pdf/1406.2696v1.pdf

[25] http://arxiv.org/pdf/1405.3297v1.pdf

[26] http://arxiv.org/pdf/hep-th/9803132v2.pdf

[27] http://arxiv.org/pdf/1404.3723v2.pdf

[28] Goldfain E., "*Dynamic Instability of the Standard Model and the Fine Tuning Problem*", Prespacetime Journal, **12**(12), (2012), pp.1175-1181.

[29] http://arxiv.org/pdf/hep-ph/9912343.pdf

[30] http://arxiv.org/pdf/0807.4313.pdf

[31] http://arxiv.org/pdf/hep-ph/0407242.pdf

[32] Goldfain E., "*Dynamics of Neutrino Oscillations and the Cosmological Constant Problem*", Prespacetime Journal **2**(3), (2011), pp.442-446.

[33] Goldfain E., "*On the asymptotic transition to complexity in quantum chromodynamics*", Comm. Nonlin. Sci. Numer. Simul. **14**(4), (2009), pp. 1431-1438.

[34] Goldfain E., "*Chaos in Quantum Chromodynamics and the Hadron Spectrum*", Electronic Journal of Theoretical Physics, **7**(23), (2010), pp. 75-84.

Section 6:

[1] http://arxiv.org/pdf/physics/0610010.pdf

[2] Tarasov V. E., *"Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media",* (2011), Springer-Verlag.

[3] see e. g., Shnir, Yakov M., "*Magnetic Monopoles"*, (2005), Springer-Verlag.

[4] Ryder L. H., "*Quantum Field Theory*", (1996), Cambridge University Press.

[5] Kaku, M., "*Quantum Field Theory, a Modern Introduction*", (1993), Oxford University Press.

Section 7:

[1] E. P. Wigner, Ann. of Math. 40, 149 (1939).

[2] M. D. Schwartz, "*Quantum Field Theory and the Standard Model*", (2014), Cambridge University Press, New York.

[3] A. Duncan, "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[4] http://arxiv.org/pdf/hep-th/0507291v2.pdf

[5] Blumenhagen R. and Plauschinn E., "*Basics in Conformal Field Theory*" in "Introduction to Conformal Field Theory", (2009), Lectures Notes in Physics, vol. 779, pp. 5-86.

[6] Goldfain E., "*Fractional dynamics and the TeV regime of field theory*", (2008), Comm. Nonlin. Science and Numer. Simul., 13, 3, pp. 666-76.

[7] Goldfain E., "*Ultraviolet Completion of Electroweak Theory on Minimal Fractal Manifolds*", Prespacetime Journal, 5(10), pp. 945-952, (2014).

[8] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", (2014), Quantum Matter, 3(3), pp. 256-263.

Section 8:

[1] http://d22izw7byeupn1.cloudfront.net/files/model_RMP.pdf

[2] http://arxiv.org/pdf/1106.5787.pdf

[3] http://www.physics.ntua.gr/cosmo11/Naxos2011/09-16%20Friday%20Talks/Scalisi.pdf

[4] http://arxiv.org/pdf/1208.1030v2.pdf

[5] http://arxiv.org/pdf/0706.3688v1.pdf

[6] http://arxiv.org/pdf/hep-th/0409096v1.pdf

[7] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[8] Goldfain E., "*Multifractal Theory and Physics of the Standard Model*", Prespacetime Journal, 5(7), (2014), pp. 554-565.

[9] Goldfain E., "*Fractal Space-time and the Dynamic Generation of Mass Scales in Field Theory*", Prespacetime Journal 5(9), (2014), pp. 843-851.

[10] Goldfain E., "*Emergence of the Electroweak Scale from Fractal Spacetime*", Prespacetime Journal, 4(9), (2013), pp. 923-926.

[11] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", Quantum Matter, 3(3), (2014), pp. 256-263.

[12] Goldfain E., "*Fractal Propagators and the Asymptotic Sectors of Quantum Field Theory*", Prespacetime Journal, 5(8), (2014), pp. 712-719.

[13] Goldfain E., "*Fractional Field Theory and Physics Beyond the Standard Model*", Prespacetime Journal, 3(5), (2012), pp. 435-438.

Section 9:

[1] Schwartz M. D., "*Quantum Field Theory and the Standard Model*", (2013), Cambridge University Press.

[2] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[3] http://arxiv.org/pdf/hep-th/0606003v2.pdf

[4] http://arxiv.org/pdf/hep-th/0612084v1.pdf

[5] http://arxiv.org/pdf/0904.1272v2.pdf

[6] Consoli M., "*The Vacuum Condensate: a Bridge from Particle Physics to Gravity?*" in "*Vision of Oneness*", (2011), Licata I. and Sakaji A. (editors), Aracne Editrice, pp. 313-330.

[7] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", (2014), Quantum Matter, 3(3), pp. 256-263.

[8] Goldfain E., "*Fractional Field Theory and Physics Beyond the Standard Model*", Prespacetime Journal, 3(5), (2012), pp. 435-438.

[9] http://arxiv.org/pdf/1210.2754v3.pdf

[10] http://arxiv.org/pdf/1107.5041v4.pdf

[11] West B. *et al.*, "*Physics of Fractal Operators*", (2003), Springer-Verlag.

[12] Luo A. C. J. and Afraimovich V. (editors), "*Long-range Interactions, Stochasticity and Fractional Dynamics",* (2010), Springer-Verlag series in *Nonlinear Physical Science.*

[13] Goldfain E., "*On a Natural Solution for the Hierarchy Problem Using Dimensional Regularization*", Prespacetime Journal, 2(3), (2011), pp. 437-439.

[14] Goldfain E., "*Fractional Field Theory and High-Energy Physics: New Developments*", (2013), in Horizons in World Physics, 279, Nova Science Publishers, pp. 69-92.

***Appendix A*:**

[1] Goldfain E., "*Dynamic Instability of the Standard Model and the Fine Tuning Problem*", Prespacetime Journal, 12(12), (2012), pp.1175-1181.

Appendix B:

[1] Goldfain E., "*Limitations of Perturbative Renormalization and the Challenges of the Standard Model*", Prespacetime Journal, 5(1), (2014), pp. 1-7.

[2] http://isites.harvard.edu/fs/docs/icb.topic1146665.files/III-9-RenormalizationGroup.pdf

Appendix C:

[1] Goldfain E., "*Multifractal Theory and Physics of the Standard Model*", Prespacetime Journal, 5(7), (2014), pp. 554-565.

Appendix D:

[1] Schwartz M. D., "*Quantum Field Theory and the Standard Model*", (2013), Cambridge University Press.

[2] Duncan A., "*Conceptual Framework of Quantum Field Theory*", (2012), Oxford University Press.

[3] http://arxiv.org/pdf/hep-th/9702027.pdf

[4] Goldfain E., "*Fractal Spacetime as Underlying Structure of the Standard Model*", (2014), Quantum Matter, 3(3), pp. 256-263.

[5] Goldfain E., "*Fractional Field Theory and High-Energy Physics: New Developments*", (2013), in Horizons in World Physics, 279, Nova Science Publishers, pp. 69-92.

[6] Goldfain E., "*Non-Equilibrium Dynamics as Source of Asymmetries in High-Energy Physics*", Electronic Journal for Theoretical Physics, (2010), 7(24), pp. 219-234.

Conclusions:

[1] https://medium.com/starts-with-a-bang/the-road-less-traveled-to-quantum-gravity-594ac38e4544

[2] Goldfain E., "*Fractional dynamics and the TeV regime of field theory*", (2008), Comm. Nonlin. Science and Numer. Simul., 13, 3, pp. 666-76.

[3] Goldfain E., "*Fractional dynamics and the Standard Model for particle physics*", (2009), Comm. in Nonlinear Science and Numer. Simul. 13(7), pp. 1397-1404.

[4] Goldfain E., "*Complexity in quantum field theory and physics beyond the standard model*", Chaos, Solitons and Fractals, (2006), 28(4), pp. 913-922.

[5] Goldfain E. and Smarandache F., "*On Emergent Physics, Un-particles and Exotic Un-matter States*", Progress in Physics, (2008), 4, pp. 10-15.

[6] Goldfain E., "*Fractional dynamics and collider phenomenology*", (2009), Comm. Nonlin. Science and Numer. Simul. 14(5), pp. 2289-2292.